ERRATUM/AMPLIFICATION FOR “K-THEORETIC TUTTE POLYNOMIALS OF MORPHISMS OF MATROIDS”

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[DES21, Remark 7.4] claimed erroneously that [DES21, Lemma 7.2] does not generalize to settings outside of the flag-geometric diagram. The claim was based on a computer computation of an example, but the function `kHpoly` in the code (found at `https://github.com/chrisweur/kTutte`) that produced the example has a bug. We thank Matt Larson for a conversation that led to the discovery of this error. No other statement in the paper is affected.

Here, we establish a generalization of [DES21, Lemma 7.2], whose proof was a variation of that of [FS12, Lemma 6.2]. The statement here is in terms of the notions developed in [BEST23]; see Section 11 of *loc. cit.* for an explanation of how the statement here recovers [DES21, Lemma 7.2] (and the generalization of [DES21, Lemma 7.2] to the “Las Vergnas diagram” setting that was erroneously claimed to be false in [DES21, Remark 7.4]).

Let M and M' be matroids on a common ground set $[n] = \{1, \dots, n\}$. Let X_n be the permutohedral variety of dimension $n-1$, on which we have K-classes $\mathcal{S}_{\mathrm{M}}$ and $\mathcal{Q}^\vee_{\mathrm{M}'}$ as defined in [BEST23]. Let $\chi : K(X_n) \to \mathbb{Z}$ be the sheaf Euler characteristic map.

Proposition 1. Suppose M is coloopless and M' is loopless. Then, we have $\chi(\bigwedge^p \mathcal{S}_{\mathrm{M}} \otimes \bigwedge^q \mathcal{Q}^\vee_{\mathrm{M}'}) = 0$ for all $p \neq q$.

Proof. Let $T = \mathbb{G}_m^{[n]}$. We show that the T-equivariant Euler characteristic $\chi^T(\bigwedge^p \mathcal{S}_{\mathrm{M}} \otimes \bigwedge^q \mathcal{Q}^\vee_{\mathrm{M}'})$ is zero. We do so via equivariant localization and the method of flipping cones, which are reviewed and extended in [DES21, Section 4]. We follow the notations laid out there.

Let Q be the Minkowski sum of the base polytopes of M and M'. Every vertex of Q is a sum $\mathbf{e}_B + \mathbf{e}_{B'}$ for a unique pair of bases B and B' of M and M' (respectively), in which case we denote $\mathrm{C}_{B,B'} := \mathrm{Cone}\{Q - \mathbf{e}_B - \mathbf{e}_{B'}\} \subset \mathbb{R}^n$. By the Atiyah–Bott localization formula [Nie74, 4.7] and Ishida’s generalization of Brion’s formula [Ish90, Theorem 2.3] (in the form of the second statement in [DES21, Theorem 4.2]), we find that

$$\varphi_{pq} := \chi^T\left(\bigwedge\nolimits^p \mathcal{S}_{\mathrm{M}} \otimes \bigwedge\nolimits^q \mathcal{Q}^\vee_{\mathrm{M}'}\right) = \sum_{\mathbf{e}_B + \mathbf{e}_{B'} \in \mathrm{Vert}(Q)} \mathrm{Hilb}(C_{B,B'}) \sum_{\mathfrak{p} \in \binom{B}{p}} \sum_{\mathfrak{q} \in \binom{[n]\setminus B'}{q}} \mathbf{t}^{-\mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}}} \in \mathbb{Q}[t_1^\pm, \dots, t_n^\pm].$$

Let P be the convex hull of the $-\mathbf{e}_{\mathfrak{p}} + \mathbf{e}_{\mathfrak{q}}$ appearing in the summation, which is contained in the cube $[-1,1]^n \subset \mathbb{R}^n$. By [FS12, Corollary 2.4], which states that the Newton polytope of φ_{pq} is contained in P, to show $\varphi_{pq} = 0$ it suffices to show that both $\varphi_{pq}|_{\mathbf{e}_\ell = -1}$ and $\varphi_{pq}|_{\mathbf{e}_\ell = 1}$ are zero for all $\ell \in [n]$, since P lies in the hyperplane $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = -p + q\}$ and $p \neq q$.

Fix an arbitrary $\ell \in [n]$. We now show $\varphi_{pq}|_{\mathbf{e}_\ell = -1} = 0$ using the method of flipping cones “in slices” as described in [DES21, Theorem 4.7]. Because M is coloopless, every vertex $\mathbf{e}_B + \mathbf{e}_{B'}$ of the face of Q minimizing in the $\mathbf{e}_\ell$ direction satisfies $\ell \notin B$. Thus, we find that if $C_{B,B'}$ is $\mathbf{e}_\ell$-pointed, then $\ell \notin B$ (see [DES21, Definition 4.3] and the comment below it). In particular, whenever $C_{B,B'}$ is $\mathbf{e}_\ell$-pointed, we have that the ℓ-th coordinate of $-\mathbf{e}_\mathsf{p} + \mathbf{e}_\mathsf{q}$ appearing in the corresponding summation is strictly larger than -1. Applying [DES21, Theorem 4.7] then yields $\varphi_{pq}|_{\mathbf{e}_\ell=-1} = 0$. The proof of $\varphi_{pq}|_{\mathbf{e}_\ell=1} = 0$ is similar, where one uses that M' is loopless. $\square$

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