Lecture notes for the course on

Representation theory

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Abstract. These are the lecture notes accompanying the lecture "Representation Theory" offered at the University of Bern in the Autumn Semester 2021. Sections will be added as the semester goes on. There are undoubtedly still many typos and errors. Please inform me of any typos/errors you find, and feel free to contact me with any other questions.

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Version history

New parts added in green
Major changes/errors in red
Medium changes/errors in orange
Minor changes/typos in black

20.09.2021	Introduction, Sections 1 and 2
22.09.2021	Typos
28.09.2021	Section 3
	In Remarks 2.4 and 2.9 replaced equiv. rel. by subspace
	Added diagram to Remark 2.5
07.10.2021	Section 4
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	Exercise 4.22
27.10.2021	Section 6.2
03.11.2021	Sections 7 and start of Section 8
	Typos in section 2 ($(-1)^{\mathrm{sgn}(\sigma)}$ should be $\mathrm{sgn}(\sigma)$).
10.11.2021	Section 8.1
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	Reformulated paragraph just before section 8.1
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	Removed Remark 6.7
15.12.2021	Section 11 final part
07.01.2022	Typo in Theorem 8.35
11.01.2022	Added Proposition 2.23, which is used in Lemma 11.9
	Added Proposition 11.12
	Typos on page 62
19.01.2022	Various typos
	References to exercise sheets
	Added a sentence in second proof of Theorem 3.26
	Small change in proof of Proposition 8.18
	Typo in Definition 8.24 ($\mathrm{col}_\lambda(b') < \mathrm{col}_\lambda(b)$)
	Incomplete sentence on page 48
	Page 53: "diagonalizable" → "diagonal"
??.01.2022	Typo in Remark 2.6

Introduction

If you ask 10 mathematicians the question "what is representation theory?", you will most likely get 10 different answers[1]. Here is my answer:

Representation theory is the study of *symmetries* of *vector spaces*.

The aforementioned symmetry is encoded by an algebraic object acting on the vector space. In this course, we will deal with *representation theory of groups*, so this algebraic object will be a group.

Scope of the course. This course can be roughly divided in two parts:

(1) *Representation theory of finite groups (over $\mathbb{C}$):* This is the canonical thing to teach in any first course on representation theory, and for good reasons. After introducing basic concepts in representation theory and proving some first important results (in particular: Maschke's theorem on complete reducibility and Schur's lemma), we will introduce a powerful tool named *character theory*. Thanks to character theory, we can basically understand everything about the representation theory of any given finite group.

(2) *Representation theory of the symmetric group and the general linear group:* Two of the most important groups are the symmetric group $\mathfrak{S}_d$, and the general linear group $GL(n, \mathbb{C})$. For any fixed d, we can already understand the representation theory of $\mathfrak{S}_d$ by using the methods of part (1). We will begin part (2) with a systematic study of the representation theory of $\mathfrak{S}_d$ for all d simultaneously; this involves some beautiful connections with algebraic combinatorics (keywords here are "Young tableau" and "Schur polynomial"). The general linear group $GL(n, \mathbb{C})$ is not a finite group, but it is deeply related to the symmetric group via *Schur-Weyl duality*, which will allow us to understand its representation theory as well.

Limitations. Representation theory is a vast subject and obviously we can only scratch the surface within the scope of any one-semester course. At the end of the course we will have seen a first glimpse of representation theory of algebraic groups (or if you want: Lie groups); a natural next step would be a general study of representation theory of algebraic groups or Lie groups (and this is indeed the next thing that is done in our main reference [**FH91**]). An important tool for this which I most likely won't be able to introduce even in the case of $GL(n, \mathbb{C})$ is the notion of a *Lie algebra*.

We will restrict ourselves to the field $\mathbb{C}$ during this course. Almost everything we do will be equally valid for other algebraically closed fields of characteristic zero, and for studying representation theory over non-algebraically closed fields one can

[1] I guess the same is true for most areas of Mathematics

get quite far by passing to the algebraic closure. On the other hand, representation theory over fields of positive characteristic (known as *modular representation theory*) is an active area of research which is far outside the scope of this course.

Finally, there are many other flavors of representation theory, like representation theory of associative algebras or of quivers, that we won't even have time to mention.

Prerequisites. The only formal prequisite for this course is a good working knowledge on linear algebra. I will also assume a little bit of familiarity with group theory: all the necessary background on groups will be recalled in the first lecture, but we will go over this quite fast.

In the current semester there is also a course on associative algebras taking place at the university of Bern. Both courses are independent from one another, but there will be some amount of overlap, in particular when we discuss the group algebra of a finite group. I am not going to assume that all participants of the Representation Theory course are also taking Associative Algebras, but I still wanted to point out this connection.

References. For part 1, we will follow Lectures 1,2,3 in [**FH91**] quite closely. Another excellent classical resource for this material is [**Ser77**]. The main reference for part 2 are lectures 4 and 6 in [**FH91**].

Chapter 10 of [**MS21**] gives a brief overview of representation theory, but sometimes without proofs. In fact, my original goal when planning this course was to cover the material in that chapter, but more in depth.

Practicalities related to the course.

- The lecture takes place every Thursday 8:15-10:00 in ExWi room B116, from September 23 up to and including December 16.
- Please remember that university regulations currently require all participants to have a valid COVID certificate in order to attend the lecture. Participants are also required to wear a mask.
- The only formal requirement to pass the course is to pass the oral exam at the end of the semester. In particular, there is no "Testatbedingung", and you are not required to attend the lecture or solve/hand in exercises (but you are of course encouraged to do so!). More info on the exam will follow later.
- There will be (non-obligatory) homework exercises every two weeks, which you can hand in via ILIAS. I will then provide feedback on your solutions. Every homework will be accompanied by an exercise session (which will occupy one half of that day's lecture, i.e. 45 minutes). The first homework will appear online at some point between lectures 1 and 2; the first exercise session will be during lecture 3. I might shift the frequency of the exercise sessions around a bit if that fits better with the course material.
- Any parts marked with a * can be safely ignored: I will not build on them further nor ask about them in the exam.
- There is a forum on ILIAS that you can use to ask any (practical or mathematical) questions related to the lecture. You can of course still ask me questions live during or after the lecture, or contact me via e-mail.

CHAPTER 1

Preliminaries

This chapter contains background material on group theory and on linear algebra that will be used throughout the course. The section on group theory essentially starts from scratch. For the section on linear algebra I will assume familiarity with basic linear algebra; the section is instead focused on introducing the *tensor product*, which we will use very often. The study of tensor products and related notions is sometimes referred to as *multilinear algebra*, hence the title of the section.

In an attempt to balance between providing enough background material to get everyone on the same page, while at the same time not getting bogged down for too long before starting with the actual content of the course, I decided to do the following: In the first lecture of the semester we will quickly go through the section on group theory, assuming that most things there are familiar from a previous course in group theory. The remaining part of the first lecture will be spent on going through the chapter on tensor products (more slowly, as I won't assume everyone is familiar with this). Independent of how far we get, in lecture 2 we will jump to Chapter 2 and start doing representation theory; the remaining background on multilinear algebra we be introduced when we need it.

Both sections might be updated later in the semester, if I realize I forgot to introduce something important.

1. A brief recap on group theory

Definition 1.1. A *group* $(G, e, *)$ consists of

- a set G,
- a chosen element $e \in G$, called *neutral element*,
- a binary operation

$$\begin{aligned} * : G \times G &\to G \\ (g, h) &\mapsto g * h \end{aligned}$$

satisfying the following axioms:

- Associativity: for all $f, g, h \in G$, we have

$$f * (g * h) = (f * g) * h,$$

- Neutral element: for all $g \in G$, we have that

$$e * g = g * e = g,$$

- Inverse element: for each $g \in G$, there exists a $g^{-1} \in G$ such that

$$g * g^{-1} = e = g^{-1} * g$$

The group is called *abelian*, if the operation is commutative, i.e. for all $g, h \in G$ we have

$$g * h = h * g.$$

Remark 1.2. As the neutral element is uniquely determined, one usually writes $(G, *)$ instead of $(G, e, *)$, or even just G if the operation is clear from the context. Additionally, when dealing with a group $G = (G, \cdot)$, we will often write gh instead of $g \cdot h$.

Definition 1.3. A *homomorphism* (or simply *morphism*) of groups $(G, *_G)$ and $(H, *_H)$ is a map $\varphi : G \to H$ respecting the multiplication: $\varphi(g_1 *_G g_2) = \varphi(g_1) *_H \varphi(g_2)$. If φ is a bijective (resp. injective/surjective), then it is called an *isomorphism* (resp. *monomorphism/epimorphism*). Note that if φ is an isomorphism, then the inverse map $\varphi^{-1} : H \to G$ is also a morphism (and hence an isomorphism). If there exists an isomorphism between G and H, we say G and H are *isomorphic*, and write $G \cong H$.

1.1. Examples.

Example 1.4. Here are some first examples of abelian groups:

- The integers $\mathbb{Z}$ form an abelian group with respect to addition.
- If $(k, +, \cdot)$ is a field, then $(k, +)$ and $(k \setminus \{0\}, \cdot)$ are abelian groups.
- For $n \in \mathbb{Z}$, we have the cyclic group $(\mathbb{Z}/n\mathbb{Z}, +)$, which is a finite abelian group. The elements of $\mathbb{Z}/n\mathbb{Z}$ are the numbers $0, \ldots, n-1$, and the operation is addition modulo n.

Example 1.5. Consider a square in the plane. The set of isometries of the plane that leave the square invariant is a non-abelian group (with group operation being composition), known as the *dihedral group* D_8. It has 8 elements: the identity, three rotations, and four reflections.

More generally, for $n \geq 3$ the set of isometries of a regular n-gon is a non-abelian group with $2n$ elements, known as the *dihedral group* D_{2n}.

Example 1.6. For $d \in \mathbb{N}$, the *symmetric group* $\mathfrak{S}_d$ is the set of all bijections from the set $[d] := \{1, 2, \ldots, d\}$ to itself, with operation given by composition. $\mathfrak{S}_d$ is a finite group, with $d!$ elements. It is customary to denote elements in $\mathfrak{S}_d$ by *cycle notation*. For instance, $(164)(25) \in \mathfrak{S}_6$ denotes the permutation that sends 1 to 6, 6 to 4, 4 to 1, 2 to 5, 5 to 2, and fixes 3.

The *sign* (or *signum*) of a permutation $\sigma \in \mathfrak{S}_d$ is defined as $\operatorname{sgn}(\sigma) := (-1)^{N(\sigma)}$, where $N(\sigma)$ is the number of pairs $(i, j) \in [d]^2$ with $i < j$ but $\sigma(i) > \sigma(j)$. It can also be defined by writing σ as a product of transpositions (ab); then $\operatorname{sgn}(\sigma) = (-1)^m$, where m is the number of transpositions.

More generally if X is a set we can consider the group $\mathfrak{S}_X$ of bijections from X to itself. If $|X| = d < \infty$ then $\mathfrak{S}_X \cong \mathfrak{S}_d$ (by identifying $|X|$ with $\{1, \ldots, d\}$).

Example 1.7. Let V be a k-vector space. The *general linear group* $GL(V)$ is the set of automorphisms of V (i.e. invertible linear maps from V to itself), with the group operation being composition. If V is n-dimensional and we choose a basis, we can identify $GL(V) =: GL(n, k)$ with the group of invertible $n \times n$ matrices. The *special linear group* $SL(n, k) \subset GL(n, k)$ is the subgroup of matrices/automorphisms with determinant one.

1.2. Conjugacy classes.

Definition 1.8. We say that two group elements $g_1, g_2 \in G$ are *conjugate* to each other if there exists a $h \in G$ such that $h^{-1}g_1h = g_2$. One can easily verify that being conjugate is an equivalence relation; the equivalence classes with respect to this equivalence relation are called *conjugacy classes*.

Example 1.9. If G is abelian, every element is only conjugate to itself. Hence the conjugacy classes are singletons.

Example 1.10. Two matrices in $GL(n,\mathbb{C})$ are conjugate if and only if they have the same Jordan normal form.

Exercise 1.11. Describe the conjugacy classes in $\mathfrak{S}_3$. What about $\mathfrak{S}_d$ for general d? (See Proposition 7.2 for the answer.)

1.3. Group actions.

Definition 1.12. Let G be a group and X be a set. A *left group action* of G on X is a map $\phi : G \times X \to X$ such that $\phi(e, x) = x$ and

$$\phi(gh, x) = \phi(g, \phi(h, x)), \tag{1.1}$$

for all $x \in X$ and $g, h \in G$. Similarily, a *right group action* of G on X is a map $\phi : G \times X \to X$ such that $\phi(e, x) = x$ and

$$\phi(gh, x) = \phi(h, \phi(g, x)), \tag{1.2}$$

for all $x \in X$ and $g, h \in G$.

For left group actions, we typically write $\phi(g, x) =: g \cdot x$ so (1.1) becomes

$$(gh) \cdot x = g \cdot (h \cdot x).$$

For right group actions, we write $\phi(g, x) =: x \cdot g$, and (1.2) becomes

$$x \cdot (gh) = (x \cdot g) \cdot h.$$

An action is called *faithful* if the only $g \in G$ that acts trivially is the neutral element:

$$\phi(g, x) = x \quad \forall x \in X \implies g = e.$$

For every $x \in X$, the set $\{\phi(g, x) \mid g \in G\} \subset X$ is called the *orbit* of x. The orbits of X form a partition of X, with two elements x and y belonging to the same orbit if and only if there is a $g \in G$ such that $g \cdot x = y$. If there is only one orbit (that is, for every x and y in X there is a $g \in G$ such that $g \cdot x = y$) we call the action *transitive*.

Example 1.13. The symmetric group $\mathfrak{S}_X$ acts on the set X by definition: for $x \in X$ and $\sigma \in \mathfrak{S}_X$ (that is, $\sigma : X \to X$ a bijection), we have $\sigma \cdot x := \phi(\sigma, x) := \sigma(x)$. This is a left action since the group operation on $\mathfrak{S}_X$ is the composition $\circ$ of maps, and by convention $\sigma \circ \sigma'$ means we first apply σ', and then apply σ.

In fact, equipping a set X with a left action of a group G is equivalent to giving a morphism $G \to \mathfrak{S}_X$. This morphism is an injection if and only the action is faithful.

Example 1.14. Every group acts on itself from the left, by $\phi(g, h) := gh$. This action is faithful and transitive. In particular, it now follows from Example 1.13 that every (finite) group can be embedded into a (finite) symmetric group.

Example 1.15. For any group G, we have another (left) action of G on itself, namely by conjugation: $\phi(g,h) = ghg^{-1}$. The orbits of this group action are the conjugacy classes.

1.4. Subgroups, normal subgroups, quotients. The notions of *subgroup* and *normal subgroup* are central in any introductory course in group theory. They will be less essential for us, as we will often just work with one fixed group.

Definition 1.16. A *subgroup* of a group $(G, \cdot)$ is a subset $H \subseteq G$ that is closed under the group action. We write $H \leq G$. For $g \in G$, we have the *left coset* $gH = \{gh \mid h \in H\}$, and the *right coset* $Hg = \{hg \mid h \in H\}$. These can also be seen as orbits of group actions, namely of the action of H on G by right (respectively left) multiplication.

If in addition $gHg^{-1} = H$ (or more precisely $h \in H, g \in G \implies ghg^{-1} \in H$, or in other words H is invariant under conjugation), then we say H is a *normal subgroup* and write $H \trianglelefteq G$. In this case the left and right cosets agree: $gH = Hg$ for every $g \in G$. The set of cosets is denoted by G/H and naturally inherits a group structure from G via $(gH) \cdot (g'H) := gg'H$. The group G/H is called the *quotient* of G and H.

2. A crash course on multilinear algebra

Here we introduce some constructions in multilinear algebra, that will appear all over the place during later lectures. You might have encountered some (or all) of them in a linear algebra course or somewhere else in mathematics. We will work over a fixed field k (which you can safely assume to be $\mathbb{C}$). Almost all vector spaces we encounter will be finite-dimensional (and even the infinite-dimensional ones will usually come equipped with a basis).

2.1. Dual vector spaces. We briefly recall the notion of dual of a vector space.

Definition 2.1. Given a vector space V, its *dual* V^* is the space $\mathrm{Hom}_k(V, k)$ of all linear forms on V.

If V is an n-dimensional vector space, then V^* is also n-dimensional, but there is no canonical way of identifying V with V^*. Given a basis $e_1, \dots, e_n$ of V, we can define the *dual basis* $e_1^*, \dots, e_n^*$ by $e_i^*(e_j) = \delta_{i,j} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$ Then $e_i \mapsto e_i^*$ gives an isomorphism between V and V^*, but it depends on the chosen basis.

Remark 2.2. If V is finite-dimensional, then there is a canonical identification between V and $(V^*)^*$, which is given by sending $v \in V$ to the map $eval_v : V^* \to k$ that sends $\beta \in V^*$ to $\beta(v)$. We will in the future always identify $(V^*)^*$ with V.

2.2. Tensor products. Given two vector spaces V and W, we can build a new vector space $V \otimes W$ out of them called the *tensor product*. There are (at least) two ways to define the tensor product: we begin with the most down-to-earth one.

Definition 2.3. Let $\{e_i\}_{i \in I}$ be a basis of V and $\{f_j\}_{j \in J}$ be a basis of W. Then $V \otimes W$ is the vector space with basis $\{e_i \otimes f_j\}_{i \in I, j \in J}$. Moreover, for every $v \in V$ and $w \in W$, we have an element $v \otimes w \in V \otimes W$ defined as follows: if $v = \sum a_i e_i$ and $w = \sum b_j f_j$, then $v \otimes w := \sum_{i,j} (a_i b_j) e_i \otimes f_j$.

Even more explicitly, if $V = k^n$ and $W = k^m$, then $V \otimes W$ is the space of $n \times m$ matrices: $e_i \otimes e_j$ is the matrix E_{ij} with a 1 on position (i, j) and 0's everywhere else, and more generally $v \otimes w$ is the product vw^T (where we interpreted v and w as column vectors).

It is important to realize that not every element of $V \otimes W$ can be written in the form $v \otimes w$ (but it can always be written as a linear combination $\sum_i v_i \otimes w_i$, in many different ways). If $V = k^n$ and $W = k^m$, the tensors of the form $v \otimes w$ are precisely the rank one matrices. We will call tensors of the form $v \otimes w$ *pure*.

From the definition, one can deduce the following calculation rules:

$$(\lambda v) \otimes w = v \otimes (\lambda w) = \lambda(v \otimes w), \tag{2.1}$$

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \tag{2.2}$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2. \tag{2.3}$$

Remark 2.4. An alternative way of defining $V \otimes W$ is as a quotient $F(V \times W)/Z$, where $F(V \times W)$ is the free vector space on the set $V \times W$ (which is the infinite-dimensional vector space with basis $\{e_{v,w} \mid v \in V, w \in W\}$), and Z is the subspace generated by the analogues of (2.1,2.2,2.3):

$$e_{\lambda v,w} - \lambda e_{v,w} \text{ and } e_{v,\lambda w} - \lambda e_{v,w}, \tag{2.4}$$

$$e_{v_1+v_2,w} - e_{v_1,w} - e_{v_2,w}, \tag{2.5}$$

$$e_{v,w_1+w_2} - e_{v,w_1} - e_{v,w_2}. \tag{2.6}$$

The identification with our previous construction is given by sending the equivalence class of $e_{v,w}$ to $v \otimes w$.

Remark 2.5. * The tensor product $V \otimes W$ comes equipped with a bilinear map

$$\begin{aligned} \phi_{V,W} : V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w. \end{aligned}$$

The pair $(V \otimes W, \phi_{V,W})$ satisfies the following universal property: suppose U is any other vector space and $\varphi : V \times W \to U$ any bilinear map. Then there is a unique linear map $\psi : V \otimes W \to U$ such that $\varphi = \psi \circ \phi_{V,W}$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{\phi_{V,W}} & V \otimes W \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \psi} \\ & & U \end{array}$$

Put briefly: giving a bilinear map from a product of vector spaces is the same as giving a linear map from their tensor product.

If $(V \tilde{\otimes} W, \tilde{\phi}_{V,W} : V \times W \to V \tilde{\otimes} W)$ were any other pair satisfying the same universal property, one can verify that there is a (unique) isomorphism $\alpha : V \otimes W \xrightarrow{\cong} V \tilde{\otimes} W$ such that $\tilde{\phi}_{V,W} = \alpha \circ \phi_{V,W}$. Because of this, we can think of the universal property as giving a *definition* of the tensor product.

Remark 2.6. Taking the tensor product is a *functorial* construction. That is to say, if we have linear maps $f : V \to V'$ and $g : W \to W'$, they induce a linear map $f \otimes g : V \otimes W \to V' \otimes W'$. On pure tensors, $f \otimes g$ is given by $(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$. This has all nice properties you might expect, like $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$.

Exercise 2.7. Consider the linear maps $f, g : \mathbb{C}^2 \to \mathbb{C}^2$ corresponding to the matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ (where we picked the standard basis $\{e_1, e_2\}$ of $\mathbb{C}^2$). Write down the matrix corresponding to the linear map $f \otimes g : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ with respect to the basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.

If $V_1, \dots, V_d$ are vector spaces we can define their tensor product $V_1 \otimes \cdots \otimes V_d$ by generalizing any of the definitions above:

Definition 2.8. Let $\{e_{j,i}\}_{i \in I_j}$ be a basis of V_j. Then $V_1 \otimes \cdots \otimes V_d$ is the vector space with basis $\{e_{1,i_1} \otimes \dots \otimes e_{d,i_d}\}_{i_j \in I_j}$. Moreover, for every $(v_1, \dots, v_d) \in V_1 \times \dots, V_d$, we have an element $v_1 \otimes \dots \otimes v_d \in V_1 \otimes \dots \otimes V_d$ defined as follows: if $v_j = \sum_i a_{j,i} e_{j,i}$, then $v_1 \otimes \dots \otimes v_d = \sum (a_{1,i_1} \cdots a_{d,i_d}) e_{1,i_1} \otimes \dots \otimes e_{d,i_d}$.

We again have the calculation rules:

$$v_1 \otimes \dots \otimes \lambda v_i \otimes \dots \otimes v_d = \lambda(v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_d), \tag{2.7}$$

$$\begin{aligned} v_1 \otimes \dots \otimes (v_i + v_i') \otimes \dots \otimes v_d =& v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_d \\ &+ v_1 \otimes \dots \otimes v_i' \otimes \dots \otimes v_d. \end{aligned} \tag{2.8}$$

Remark 2.9. (This is just the analogue of Remark 2.4, there is probably no point in reading this.) The tensor product $V_1 \otimes \cdots \otimes V_d$ can also be defined as a quotient $F(V_1 \times \cdots \times V_d)/\sim$, where $F(V_1 \times \cdots \times V_d)$ is the free vector space on the set $V_1 \times \cdots \times V_d$ (which is the infinite-dimensional vector space with basis $\{e_{v_1,\dots,v_d} \mid v_i \in V_i\}$), and $\sim$ is the subspace generated by the analogues of (2.7,2.8):

$$e_{v_1,\dots,\lambda v_i,\dots,v_d} - \lambda e_{v_1,\dots,v_i,\dots,v_d}, \tag{2.9}$$

$$e_{v_1,\dots,v_i+v_i',\dots,v_d} - e_{v_1,\dots,v_i,\dots,v_d} - e_{v_1,\dots,v_i',\dots,v_d}. \tag{2.10}$$

The identification with our previous construction is given by sending the equivalence class of $e_{v_1,\dots,v_d}$ to $v_1 \otimes \cdots \otimes v_d$.

Remark 2.10. * The universal property from Remark 2.5 also holds, with the bilinear map $\phi_{V,W}$ replaced by the *multilinear* (that is, linear in every argument) map

$$\begin{aligned} \phi_{V_1,\dots,V_d} : V_1 \times \dots \times V_d &\to V_1 \otimes \dots \otimes V_d \\ (v_1, \dots, v_d) &\mapsto v_1 \otimes \dots \otimes v_d. \end{aligned}$$

The universal property then says that $\phi_{V_1,\dots,V_d}$ is the universal multilinear map from $V_1 \times \dots \times V_d$. If you really want to you can spell out the details as an exercise.

Remark 2.11. Taking the tensor product is commutative, associative, and distributive. That means that we have natural isomorphisms

$$\begin{aligned} V \otimes W &\cong W \otimes V \\ v \otimes w &\mapsto w \otimes v, \end{aligned}$$

$$\begin{aligned} U \otimes (V \otimes W) &\cong U \otimes V \otimes W \cong (U \otimes V) \otimes W \\ u \otimes (v \otimes w) &\mapsto u \otimes v \otimes w \mapsto (u \otimes v) \otimes w, \end{aligned}$$

$$\begin{aligned} U \otimes (V \oplus W) &\cong (U \otimes V) \oplus (U \otimes W) \\ u \otimes (v, w) &\mapsto (u \otimes v, u \otimes w). \end{aligned}$$

Remark 2.12. To avoid confusion about the symbols $\times$ and $\oplus$: for V and W vector spaces, both $V \times W$ and $V \oplus W$ mean the same thing, namely the set $\{(v,w) \mid v \in V, w \in W\}$, which is naturally a vector space of dimension $\dim V + \dim W$. We usually write $V \times W$ when talking about bilinear maps, and $V \oplus W$ in all other cases.

We probably won't encounter this, but just to point this out: when dealing with *infinite* direct product and sums, $\bigtimes_{i\in I} V_i$ is no longer the same as $\bigoplus_{i\in I} V_i$: the former is the set of all tuples $(v_i)_{i\in I}$, the latter is the set of all such tuples for which only finitely many v_i are nonzero.

Example 2.13. We have a natural isomorphism

$$V^* \otimes W \cong \mathrm{Hom}_k(V,W)$$
$$\beta \otimes w \mapsto \left(\begin{smallmatrix} V \to W \\ v \mapsto \beta(v)\cdot w \end{smallmatrix}\right),$$

where $\mathrm{Hom}_k(V,W)$ denotes the space of linear maps from V to W.

Exercise 2.14. Let V be a finite-dimensional vector space.

(1) Construct a natural linear map $V^* \otimes V \to k$.
(2) Together with the isomorphism from Example 2.13, we get a linear map $\mathrm{Hom}_k(V,V) \to k$. This is known as the *trace*. Show that this is the same as the trace you know from linear algebra.

2.3. Symmetric and exterior powers. In this section, we assume our field k has characteristic 0. For any vector space V, one can consider the d-th tensor power

$$V^{\otimes d} = \underbrace{V \otimes \ldots \otimes V}_{d \text{ times}}.$$

There is natural right action of $\mathfrak{S}_d$ on $V^{\otimes d}$ defined by

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \tag{2.11}$$

and linearly extending (that is: for $w_1, \ldots, w_m$ pure tensors, we have $(w_1 + \ldots + w_m)\cdot\sigma := w_1\cdot\sigma + \ldots + w_d\cdot\sigma)$.

Exercise 2.15. Show that the action defined above is indeed a right action (and not a left action).

The d'th symmetric power S^dV and d'th exterior power $\bigwedge^d V$ can be defined in two equivalent ways: either as a subspace of $V^{\otimes d}$, or as a quotient of $V^{\otimes d}$. We will here take the subspace approach, with the quotient approach being a property.

Definition 2.16. The *symmetric power* $S^dV \subset V^{\otimes d}$ is the subspace of all tensors $w \in V^{\otimes d}$ that are invariant under the group action (2.11):

$$w \in S^dV \iff \forall \sigma \in \mathfrak{S}_d : w\cdot\sigma = w.$$

The *exterior power* $\bigwedge^d V \subset V^{\otimes d}$ is the subspace of all tensors $w \in V^{\otimes d}$ that are *anti-invariant* under the group action (2.11):

$$w \in \bigwedge^d V \iff \forall \sigma \in \mathfrak{S}_d : w\cdot\sigma = \mathrm{sgn}(\sigma)w.$$

Remark 2.17. In case $d = 2$ and $\dim V = n$ with a chosen basis, we can view $V^{\otimes 2}$ as the space of $n \times n$ matrices. Then S^2V is the space of symmetric matrices and $\bigwedge^2 V$ is the space of skew-symmetric matrices.

Remark 2.18. There are natural projection maps $V^{\otimes d} \to S^d V$ and $V^{\otimes d} \to \bigwedge^d V$, given by the *symmetrization map*

$$\begin{aligned} V^{\otimes d} &\twoheadrightarrow S^d V \\ v_1 \otimes \ldots \otimes v_d &\mapsto v_1 \bullet \cdots \bullet v_d := \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \end{aligned} \tag{2.12}$$

and the *antisymmetrization map*

$$\begin{aligned} V^{\otimes d} &\twoheadrightarrow \bigwedge^d V \\ v_1 \otimes \ldots \otimes v_d &\mapsto v_1 \wedge \cdots \wedge v_d := \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}. \end{aligned} \tag{2.13}$$

(The factors $\frac{1}{d!}$ are needed so that the compositions $S^d V \hookrightarrow V^{\otimes d} \twoheadrightarrow S^d V$ and $\bigwedge^d V \hookrightarrow V^{\otimes d} \twoheadrightarrow \bigwedge^d V$ are both the identity.)

The kernel K of the symmetrization map is given by

$$\begin{aligned} K &= \operatorname{Span}\{v_1 \otimes \ldots \otimes v_d - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)} \mid v_1, \ldots, v_d \in V, \sigma \in \mathfrak{S}_d\} \\ &= \operatorname{Span}\{v_1 \otimes \ldots \otimes v_d - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)} \mid v_1, \ldots, v_d \in V, \sigma \in \mathfrak{S}_d \text{ a transposition}\} \end{aligned}$$

hence we get an isomorphism $S^d V \cong V^{\otimes d}/K$. In the literature, this is often taken as the definition[1] of $S^d V$. Similarily, the kernel K' of the antisymmetrization map is given by

$$\begin{aligned} K' &= \operatorname{Span}\{v_1 \otimes \ldots \otimes v_d - \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)} \mid v_1, \ldots, v_d \in V, \sigma \in \mathfrak{S}_d\} \\ &= \operatorname{Span}\{v_1 \otimes \ldots \otimes v_d + v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)} \mid v_1, \ldots, v_d \in V, \sigma \in \mathfrak{S}_d \text{ a transposition}\} \\ &= \operatorname{Span}\{v_1 \otimes \ldots \otimes v_d \mid v_1, \ldots, v_d \in V \text{ and 2 of the } v_i \text{ are equal}\}, \end{aligned}$$

and we get $\bigwedge^d V \cong V^{\otimes d}/K'$.

Remark 2.19. * The symmetric and exterior power satisfy a universal property: we call a multilinear map $\beta : V \times \ldots \times V \to W$ *symmetric*, if

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(d)}) = \beta(v_1, \ldots, v_d) \quad \forall v_1, \ldots, v_d \in V \text{ and } \sigma \in \mathfrak{S}_d,$$

and *alternating* if

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(d)}) = \operatorname{sgn}(\sigma)\beta(v_1, \ldots, v_d) \quad \forall v_1, \ldots, v_d \in V \text{ and } \sigma \in \mathfrak{S}_d.$$

Note that β is alternating if and only if

$$\beta(v_1, \ldots, v_d) = 0 \text{ when two of the } v_i \text{ are equal.}$$

Then we have a symmetric multilinear map

$$\begin{aligned} \phi : V \times \ldots \times V &\to S^d V \\ (v_1, \ldots, v_d) &\mapsto v_1 \bullet \ldots \bullet v_d \end{aligned}$$

which is universal, in the sense that any other symmetric multilinear map $\varphi : V \times \ldots \times V \to W$ factors over ϕ: there is a unique linear map $\psi : S^d V \to W$ such that $\varphi = \psi \circ \phi$.

[1] When working over positive characteristic, this is in fact the correct definition.

Similarily, we have an alternating multilinear map

$$\begin{aligned} \phi : V \times \ldots \times V &\to \bigwedge^d V \\ (v_1, \ldots, v_d) &\mapsto v_1 \wedge \ldots \wedge v_d \end{aligned}$$

which is universal, in the sense that any other alternating multilinear map $\varphi : V \times \ldots \times V \to W$ factors over ϕ: there is a unique linear map $\psi : \bigwedge^d V \to W$ such that $\varphi = \psi \circ \phi$.

If V is a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$, then a basis of $S^d V$ is given by

$$\{e_{i_1} \bullet \ldots \bullet e_{i_d} \mid 1 \leq i_1 \leq \ldots \leq i_d \leq n\}.$$

In other words, $S^d V$ is the ring of homogeneous polynomials of degree d in the variables $e_1, \ldots, e_n$. In particular we have $\dim S^d V = \binom{n+d-1}{d}$. Similarily, a basis of $\bigwedge^d V$ is given by

$$\{e_{i_1} \wedge \ldots \wedge e_{i_d} \mid 1 \leq i_1 < \ldots < i_d \leq n\}.$$

In particular we have $\dim \bigwedge^d V = \binom{n}{d}$. Note that $\bigwedge^d V = 0$ when $d > n := \dim V$.

Remark 2.20. Given a linear map $f : V \to W$, it induces linear maps

$$\begin{aligned} S^d f : S^d V &\to S^d W \\ v_1 \bullet \ldots \bullet v_d &\mapsto f(v_1) \bullet \ldots \bullet f(v_d) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \bigwedge^d f : \bigwedge^d V &\to \bigwedge^d W \\ v_1 \wedge \ldots \wedge v_d &\mapsto f(v_1) \wedge \ldots \wedge f(v_d). \end{aligned} \tag{2.15}$$

These are just restrictions of the map $f^{\otimes d}$ from Remark 2.6.

Exercise 2.21. As an interesting special case, let $\dim V = n$, take $f : V \to V$, and consider $\bigwedge^n f : \bigwedge^n V \to \bigwedge^n V$. Since $\dim \bigwedge^n V = \binom{n}{n} = 1$, this map is given by multiplication by a scalar. This scalar is known as the *determinant* of f. Show that this is the same as the determinant you know from linear algebra.

Remark 2.22. Clearly we have $S^d V \cap \bigwedge^d V = 0$. In the case $d = 2$ we have an equality

$$V^{\otimes 2} = S^2 V \oplus \bigwedge^2 V, \tag{2.16}$$

for instance because $n^2 = \binom{n+1}{2} + \binom{n}{2}$. This is just the fact that every square matrix is the sum of a symmetric matrix and a skew-symmetric matrix (in a unique way).

The obvious generalization of (2.16) is false: for $d \geq 3$ (and $n \geq 2$), the space $S^d V \oplus \bigwedge^d V$ is a strict subspace of $V^{\otimes d}$. We will see the correct generalization of (2.16) much later in the course, when dealing with *Schur functors*.

Proposition 2.23. ** The symmetric power $S^d V$ is linearly spanned by the vectors of the form $v^d := v \otimes \cdots \otimes v$.*

Proof. This follows from the *polarization identity*

$$d! \cdot v_1 \bullet \cdots \bullet v_d = \sum_{S \subseteq [d]} (-1)^{d-|S|} \left(\sum_{i \in S} v_i \right)^d . \tag{2.17}$$

In order to show (2.17), we expand the right hand side as a linear combination

$$\sum_{i_1 \leq \ldots \leq i_d} a_{i_1 \ldots i_d} v_{i_1} \bullet \cdots \bullet v_{i_d}.$$

If some j doesn't occur in $\{i_1, \ldots, i_d\}$, the coefficient $a_{i_1 \ldots i_d}$ is zero, since for every $S \not\ni j$, the contribution of $\left(\sum_{i \in S} v_i\right)^d$ cancels against the contribution of $\left(\sum_{i \in S \sqcup j} v_i\right)^d$. So the only nonzero coefficient is $a_{1 \ldots d}$; and we have $a_{1 \ldots d} = d!$ since $\left(\sum_{i \in S} v_i\right)^d$ does not have a summand $v_{i_1} \bullet \cdots \bullet v_{i_d}$ unless $S = [d]$. □

CHAPTER 2

Representation theory of finite groups

3. Introduction and first results

3.1. Fundamental definitions.

Definition 3.1. A *representation* of a group G is a vector space V (over a field k), together with a left group action

$$\begin{aligned} G \times V &\to V \\ (g, v) &\mapsto g \cdot v \end{aligned} \tag{3.1}$$

which is *linear*, i.e. for all $g \in G$, $\lambda, \mu \in k$, and $v, w \in V$:

$$g \cdot (\lambda v + \mu w) = \lambda(g \cdot v) + \mu(g \cdot v). \tag{3.2}$$

The condition (3.2) means that for every $g \in G$, the induced bijection $V \to V : v \mapsto g \cdot v$ is a linear map, hence an element of $GL(V)$. So we can restate the above definition as follows:

Definition 3.2. A *representation* (V, ρ) of a group G is a homomorphism of groups $\rho : G \to GL(V)$, where V is a vector space over some field k.

To state this once more explicitly: given a representation " $\cdot$ " $: G \times V \to V$ according to Definition 3.1, we can construct a map $\rho : G \to GL(V)$ by putting $\rho(g)(v) := g \cdot v$. This ρ is a morphism of groups since " $\cdot$ " is a group action, and hence we have a representation according to Definition 3.2. Conversely, given a representation $\rho : G \to GL(V)$ according to Definition 3.2, one verifies that $g \cdot v := \rho(g)(v)$ defines a linear group action, and hence we get a representation according to Definition 3.1.

In the future, we will typically just write "V is a representation of G" and leave the extra data (given by (3.1) or equivalently by the morphism ρ) implicit.

Definition 3.3. A *morphism* of G-representations (V, ρ_V) and (W, ρ_W), also called *G-linear map*, is a linear map $\varphi : V \to W$ that is compatible with the group action: $\varphi(g \cdot v) = g \cdot \varphi(v)$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ {\scriptstyle \rho_V(g)}\downarrow & & \downarrow{\scriptstyle \rho_W(g)} \\ V & \xrightarrow{\varphi} & W \end{array}$$

We will denote the vector space of G-linear maps from V to W by $\mathrm{Hom}_G(V, W)$.

Definition 3.4. For every group G, we have the *zero representation* $V = 0$, and the *trivial representation* $V = k$, with group action given by $g \cdot v = v$.

Examples 3.5.

- As a "0-th example", note that if G is the trivial group, then representations of G are just vector spaces, and morphisms of representations are linear maps.

- The dihedral group D_{2n} acts on $\mathbb{R}^2$ by construction; this gives a 2-dimensional representation of D_{2n} (over the real numbers).
- The symmetric group $\mathfrak{S}_d$ has a one-dimensional representation given by the signum: $V = \mathbb{C}$, and the action is given by $\sigma \cdot v = \mathrm{sgn}(\sigma)v$. This is known as the *alternating representation* or *sign representation*.

Convention 3.6. From now on, all representations we work with will be finite-dimensional representations over $\mathbb{C}$ (unless explicitly stated otherwise).

Remark 3.7. * In order to be able to give geometrically motivated examples like the second one in Examples 3.5, the following remark is in order: *every (n-dimensional) real representation naturally gives a (n-dimensional) complex representation.* If you don't mind choosing coordinates, the easiest way to think about this is to view the real representation as a map $G \to GL(n, \mathbb{R})$. The associated complex representation is then given by embedding $GL(n, \mathbb{R})$ into $GL(n, \mathbb{C})$ (every real matrix is in particular a complex matrix). If you *do* mind choosing coordinates: what we are really doing here is noting that a linear action on a real vector space V gives a linear action on the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$. The same works for any field extension.

Definition 3.8. A *subrepresentation* of a G-representation V is a linear subspace W of V that is invariant under the action of G (i.e. $g \cdot w \in W$ for all $g \in G$ and $w \in W$). This restricted action makes W into a G-representation as well, and the inclusion $W \hookrightarrow V$ is a morphism of representations.

Exercise 3.9. If $\varphi : V \to W$ is a morphism of G-representations, then $\operatorname{im}\varphi$ is a subrepresentation of W, and $\ker\varphi$ is a subrepresentation of V.

Definition 3.10. The *direct sum* of 2 G-representations V and W is the vector space $V \oplus W$, with the group action given by $g \cdot (v, w) := (g \cdot v, g \cdot w)$. In this case V and W are naturally subrepresentations of $V \oplus W$.

Remark 3.11. For vector spaces $V_1, \ldots, V_k$, $W_1, \ldots, W_\ell$, we have a natural isomorphism[1]

$$\mathrm{Hom}_{\mathbb{C}}(\bigoplus_i V_i, \bigoplus_j W_j) \cong \bigoplus_{i,j} \mathrm{Hom}_{\mathbb{C}}(V_i, W_j).$$

This isomorphism preserves compatibility with a group action. Hence for $V_1, \ldots, V_k$, $W_1, \ldots, W_\ell$ representations of G, we have

$$\mathrm{Hom}_G(\bigoplus_i V_i, \bigoplus_j W_j) \cong \bigoplus_{i,j} \mathrm{Hom}_G(V_i, W_j).$$

Definition 3.12. For any representation V, we define the *fixed point set*

$$V^G := \{v \in V \mid g \cdot v = v \quad \forall g \in G\}.$$

This is a subrepresentation of V, and we have $V^G \cong \mathbb{C}^{\oplus d}$ as G-representations, where $d := \dim V^G$ (and $\mathbb{C}$ denotes the trivial representation). Conversely, we have $v \in V^G$ if and only if $\langle v \rangle \subseteq V$ is a subrepresentation isomorphic to the trivial representation.

Definition 3.13. A G-representation V is *irreducible* if the only subrepresentations are 0 and V.

[1] This is basically the universal property of the direct sum.

Irreducible representations (sometimes abbreviated to "*irreps*") can be thought of as the "basic building blocks" of every representation. One of our main goals of this chapter is for a given finite group, to classify all irreducible representations.

3.2. Constructions. We introduce several important constructions which construct new representations out of old ones; and in addition introduce the notion of permutation representation associated to a group action.

Definition 3.14. The *tensor product* of 2 G-representations is the vector space $V \otimes W$, with the group action given by $g \cdot (v \otimes w) := g \cdot v \otimes g \cdot w$.

Remark 3.15. In particular, $V^{\otimes n}$ is a G-representation. One verifies that $S^n V$ and $\bigwedge^n V$ are subrepresentations.

Definition 3.16. The *dual* V^* of a G-representation V is again a G-representation, via the rule $g \cdot \beta(v) := \beta(g^{-1} \cdot v)$ (for $g \in G$, $\beta \in V^*$, and $v \in V$).

Exercise 3.17. (See also Exercise 1 on Sheet 1.) If you are not surprised by the appearance of "g^{-1}" above, you can probably skip this.

- Verify that this is indeed a G-representation. Where does your proof fail if we had put g instead of g^{-1}?
- Verify that for $g \in G$, $v \in V$ and $\beta \in V^*$ we have $\langle g \cdot \beta, g \cdot v \rangle = \langle \beta, v \rangle$. (Where $\langle \beta, v \rangle$ is just evaluating the map β at the vector v.)
- Let $V = \mathbb{C}^n$, so we can view our representation as a map $G \to GL(n, \mathbb{C})$. What is then the dual representation (as a map $G \to GL(n, \mathbb{C})$)?

Definition 3.18. Given two G-representations V, W, the space $\mathrm{Hom}_k(V, W)$ becomes a G-representation via $\mathrm{Hom}_k(V, W) \cong V^* \otimes W$.

Exercise 3.19. (See also Exercise 1 on Sheet 1.) Convince yourself of the following:

- For $\varphi \in \mathrm{Hom}_k(V, W)$, the map $g \cdot \varphi$ is given by $(g \cdot \varphi)(v) = g \cdot \varphi(g^{-1} \cdot v)$. Note that in the case $W = \mathbb{C}$ is the trivial representation, we recover the definition of the group action on V^*.
- $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_k(V, W)^G$. In words: the subspace $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_k(V, W)$ is a G-subrepresentation, consisting of all elements $\varphi \in \mathrm{Hom}_k(V, W)$ that are fixed under the action of G we just defined.

Definition 3.20.

- Suppose X is a finite set and G acts on X from the left. We can make the free vector space on X (which is just the vector space with basis $\{e_x : x \in X\}$) into a representation via $g \cdot e_x = e_{g \cdot x}$ and extending linearly. This is called a *permutation representation.*
- In the special case where $X = G$ and the action is by left multiplication, the representation described above is known as the *regular representation* of G, and is denoted by R_G.

Example 3.21. $\mathfrak{S}_3$ naturally acts on the set $[3] = \{1, 2, 3\}$. This gives rise to a linear action of $\mathfrak{S}_3$ on $\mathbb{C}^3$, permuting the basis vectors. Note that this permutation representation is not irreducible. Indeed: the vector $e_1 + e_2 + e_3$ is invariant under the group action and hence spans a (trivial) one-dimensional subrepresentation.

Exercise 3.22. Prove that the following two representations are isomorphic to R_G (and hence each give an alternative definition of the regular representation):

- The space $\mathrm{Map}(G, \mathbb{C})$ of maps $G \to \mathbb{C}$, with the group action given by $(g \cdot \alpha)(h) := \alpha(g^{-1}h)$ (for $\alpha \in \mathrm{Map}(G, \mathbb{C})$).

- Same as above, but with group action given by $(g \cdot \alpha)(h) := \alpha(hg)$.

Exercise 3.23. (Exercise 3 on Sheet 1) Let $\rho : G \to GL(V)$ be a representation of a finite group. Prove that every matrix $\rho(g) \in GL(V)$ is diagonalizable.

3.3. Schur's lemma and complete reducibility. In this section we prove two first results on representations of finite groups: Schur's lemma, and complete reducibility (also known as Maschke's theorem). Together, they tell us that in order to understand the representation theory of a finite group, it suffices to describe its irreducible representations.

We begin with Schur's lemma, which tells us that morphisms between irreducible representations are easy to describe.

THEOREM 3.24 (Schur's lemma). *Let G be any group.*

(1) If $\varphi : V \to W$ is a morphism of irreps, then either $\varphi = 0$ or φ is an isomorphism.
(2) For V an irrep, $\mathrm{Hom}_G(V, V) = \{\lambda \cdot id \mid \lambda \in \mathbb{C}\}$.

PROOF. (1) This follows from the fact that for $\varphi : V \to W$ a morphism of representations, $\ker(\varphi) \subseteq V$ and $\mathrm{im}(\varphi) \subseteq W$ are subrepresentations. Precisely: let $\varphi \neq 0$. Then $\ker(\varphi) \subsetneq V$, so (since V irreducible) $\ker(\varphi) = 0$; and $0 \neq \mathrm{im}(\varphi) \subseteq W$, so (since W irreducible) $\mathrm{im}(\varphi) = W$. But this means that φ is an isomorphism.

(2) Let $\varphi \in \mathrm{Hom}_G(V, V)$, and let $\lambda \in \mathbb{C}$ be an eigenvalue[2] of the linear map φ. Then $\varphi - \lambda \cdot id$ has nonzero kernel, so by (1) $\varphi - \lambda \cdot id = 0$, hence $\varphi = \lambda \cdot id$.

□

Exercise 3.25. Deduce that for V and W irreps,

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \text{(as } G\text{-representations)}, \\ 0 & \text{else.} \end{cases}$$

The next theorem, complete reducibility for complex representations of finite groups, is one of the central results of this chapter. In contrast to Schur's lemma, a version of which can be expected whenever one has a notion of "irreducible object", complete reducibility really makes us of our assumptions that G is a finite group, and that the field we work in has characteristic zero[3].

THEOREM 3.26 (Complete reducibility). *Let G be a* **finite** *group, V a representation, and $W \subset V$ a subrepresentation. Then there is another subrepresentation $W' \subset V$ such that $V = W \oplus W'$.*

FIRST PROOF. Let H_0 be any positive-definite Hermitian form on V, and define

$$H(v, w) = \sum_{g \in G} H_0(gv, gw).$$

Note that H is G-equivariant: $H(gv, gw) = H(v, w)$ for all $v, w \in V$ and $g \in G$. Take $W' = W^{\perp} := \{v \in V \mid H(v, w) = 0 \quad \forall w \in W\}$ the orthogonal complement with respect to this Hermitian form. Then $V = W \oplus W'$, and W' is G-invariant by the equivariance of H. □

[2] We are here using that $\mathbb{C}$ is an algebraically closed field.

[3] In fact, our second proof can be generalized to the case where $\mathrm{char}(k)$ does not divide $|G|$. But if $\mathrm{char}(k)$ does divide $|G|$ then complete reducibility fails.

SECOND PROOF. Take any projection map $\pi_0 : V \to W$ (i.e. pick a vector space complement U of W in V, and let $\pi_0 : V \cong U \oplus W \to W$). Define

$$\pi(v) = \sum_{g \in G} g \cdot \pi_0(g^{-1} \cdot v),$$

then one verifies that $\pi(w) = |G| \cdot w$ for $w \in W$, and that $\pi(h \cdot v) = h \cdot \pi(v)$ for all $h \in G$ and $v \in V$. Hence $\pi : V \to W$ is a surjective morphism of G-representations. Then $W' := \ker(\pi) \subset V$ is a subrepresentation, and $V = W \oplus W'$. □

Exercise 3.27. In Example 3.21, we found a subrepresentation $\mathbb{C} \cong \langle e_1 + e_2 + e_3 \rangle =: W$ of the permutation representation $\mathbb{C}^3 =: V$ of $\mathfrak{S}_3$. So by complete reducibility, there exists a complementary subrepresentation $W' \subset V$. Find this W'. Is it irreducible?

We can reformulate the theorem as follows:

Corollary 3.28 (Maschke's theorem)**.** *Every representation of a finite group is a direct sum of irreducible representations.*

Exercise 3.29. Deduce Corollary 3.28 from Theorem 3.26.

Remark 3.30. * For the people taking the course on associative algebras, here is one more reformulation of Theorem 3.26: *The group algebra* $\mathbb{C}[G]$ *of a finite group* G *is a semisimple algebra.*

Exercise 3.31. (Exercise 4 on Sheet 1.) This exercise shows that without the assumption that G is finite, Maschke's theorem fails: let $G = \mathbb{Z}$ and consider the 2-dimensional representation $\mathbb{Z} \to GL(2, \mathbb{C}) : a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Convince yourself that this is actually a representation, then find a subrepresentation that has no complement.

Remark 3.32. * Let $\mathcal{C}$ be either the category of finite groups, or the category of finite-dimensional representations of a fixed finite group G over a fixed field k, or the category of finite-dimensional A-modules of a fixed finite-dimensional k-algebra A. In each of these situations, we have a notion of "irreducible object". Moreover, one can break each object X into its irreducible parts by writing down a composition series[4]. These irreducible parts are known as the *composition factors*[5] of X.

The natural question of trying to classify all objects in our category now splits into two subquestions:

(1) What are the basic building blocks (irreducible objects)?
(2) The *extension problem*: Given two objects Y and Z, in which ways can I put them together to an object X? (I.e. find all X such that $Y \hookrightarrow X$ with $Z \cong X/Y$, this is known as an *extension* of Z by Y.)

We always have a trivial way of putting Y and Z together: namely to the direct sum $Y \oplus Z$. Maschke's theorem now can be rephrased as follows:

[4]Or more intuitively: if X has a subobject Y we can imagine that X splits into Y and X/Y. We can then split up Y and X/Y in a similar fashion and keep repeating until we are left with a list of irreducible objects.

[5]The fact that the composition factors of X are uniquely determined is known as the *Jordan-Holder theorem*. In our setting this is an immediate corollary of Corollary 3.34, but it also holds is more general settings where we don't have complete reducibility.

For representations of a finite group over $\mathbb{C}$, the question (2) above is trivial: the only extension of Z by Y is the direct sum $Y \oplus Z$.

Convention 3.33. From now on, we will always assume that our given group G is finite, unless explicitly stated otherwise.

Together, Schur's lemma and Maschke's theorem tell us that if we understand the irreducible representations of a finite group G, we can describe all its representations, and even the space of morphisms between two representations. More precisely, we have the following:

Corollary 3.34. *Let V be a representation of a finite group G.*

(1) There is a decomposition

$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k} \tag{3.3}$$

into irreducible representations (where $V_i \ncong V_j$ if $i \neq j$).

(2) The occurring irreps V_i and their multiplicities a_i are uniquely determined.

(3) The subrepresentations $V_i^{\oplus a_i} \subset V$ (called isotypic components*) are also uniquely determined.*

(4) Let W be another representation, and

$$W \cong V_1^{\oplus b_1} \oplus \cdots \oplus V_k^{\oplus b_k} \tag{3.4}$$

be its decomposition into irreducibles[6]. Then we have isomorphisms

$$\operatorname{Hom}_G(V, W) \cong \bigoplus_i \operatorname{Hom}_G(V_i^{\oplus a_i}, V_i^{\oplus b_i}) \tag{3.5}$$

$$\cong \bigoplus_i \operatorname{Mat}(a_i \times b_i, \mathbb{C}). \tag{3.6}$$

Proof. (1) is just a reformulation of Maschke's theorem. We next prove (4): by Remark 3.11, we have

$$\operatorname{Hom}_G(V, W) \cong \bigoplus_{i,j} \operatorname{Hom}_G(V_i^{\oplus a_i}, V_j^{\oplus b_j})$$

and we can write

$$\operatorname{Hom}_G(V_i^{\oplus a_i}, V_j^{\oplus b_j}) \cong \operatorname{Hom}_G(V_i, V_j)^{\oplus a_i b_j} \cong \begin{cases} 0 & \text{if } i \neq j \\ \operatorname{Mat}(a_i \times b_i, \mathbb{C}) & \text{if } i = j \end{cases}$$

by Remark 3.11 and Schur's lemma.

For (2), note that in (3.5) the left hand side is an isomorphism if and only if every summand on the right hand side is an isomorphism, which can only happen if $a_i = b_i$ for every i. Applying this to the identity map on V yields the desired uniqueness.

For (3), what we have to show is that if we have two isomorphisms

$$V \xrightarrow[\cong]{\varphi, \psi} V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

then their composition $\varphi \circ \psi^{-1}$ maps every summand $V_i^{\oplus a_i}$ to itself. This follows immediately from (3.5). □

[6]We can without loss of generality assume that the irreps appearing in (3.3) are the same as the irreps appearing in (3.4), by having some a_i and b_i equal to 0.

Remark 3.35. The decomposition of $V_i^{\oplus a_i}$ into irreducible subrepresentations is not unique. Consider for instance G the trivial group, whose only irrep is $\mathbb{C}$. Then decomposing a vector space V as a direct sum of copies of $\mathbb{C}$ amounts to choosing a basis (up to scaling of the basis vectors), which is of course not unique.

3.4. Representation theory of abelian groups.

Proposition 3.36. *Let V be an irreducible representation of an abelian group G. Then V is one-dimensional.*

Proof. The key point is that since G is abelian, it follows that for every $g \in G$ the linear map $\varphi_g : V \to V$ given by $\varphi_g(v) = g \cdot v$ is a morphism of representations. So by Schur's lemma $g \cdot v$ is a scalar multiple of v, for every $g \in G$ and $v \in V$. But if we now fix a $v \in V$, this means that the one-dimensional subspace spanned by v is a subrepresentation, which is equal to all of V since V is irreducible. □

In other words, irreps of an abelian group G are given by elements of the *dual group* $\mathrm{Hom}(G, \mathbb{C}^*)$.

Exercise 3.37. (Exercise 5 on Sheet 1.) Show that for a finite abelian group G, there is an isomorphism $G \cong \mathrm{Hom}(G, \mathbb{C}^*)$ of groups. (Hint: by the classification of finite abelian groups, G is a product of cyclic groups.) Conclude that the number of nonisomorphic irreps of G is equal to $|G|$.

Remark 3.38. An alternative proof of Proposition 3.36 can be obtained by combining Exercise 3.23 with the linear algebra fact that commuting diagonalizable matrices are simultaneously diagonalizable.

4. Character theory

Character theory is a remarkably effective tool for understanding the representations of a given finite group G. In this section we define an invariant for representations of finite groups, known as the *character*. The main points are the following:

- The character of a representation is easy to compute.
- Every representation is uniquely determined by its character (Corollary 4.13).
- Once we know the characters of all the irreps of G, we can read the decomposition of any representation into irreps off from its character (Corollary 4.14).

Another important result that follows from character theory is Theorem 4.18, which states that the number of irreps of G is equal to the number of conjugacy classes.

4.1. Characters.

Definition 4.1. Let $\rho : G \to GL(V)$ be a representation. The *character* of G is the map $\chi_V : G \to \mathbb{C}$ sending a group element g to the trace $\mathrm{tr}(\rho(g))$ of the corresponding matrix.

Remark 4.2.
(1) Since $\rho(e)$ is the identity matrix, we have $\chi_V(e) = \dim V$.
(2) For $g, h \in G$, we have $\chi_V(h^{-1}gh) = \chi_V(g)$, i.e. χ_G is constant on conjugacy classes. We call such a function $\alpha : G \to \mathbb{C}$ satisfying $\alpha(h^{-1}gh) = \alpha(g)$ a *class function*. The set $\mathbb{C}_{\mathrm{class}}(G)$ of all class functions on G is a $\mathbb{C}$-vector space, whose dimension is equal to the number of conjugacy classes of G.

Exercise 4.3. $\alpha : G \to \mathbb{C}$ is a class function if and only if $\alpha(gh) = \alpha(hg)$ for all $g, h \in G$.

Proposition 4.4. *For V and W representations of G, we have:*

$$\chi_{V\oplus W} = \chi_V + \chi_W, \tag{4.1}$$

$$\chi_{V\otimes W} = \chi_V \cdot \chi_W, \tag{4.2}$$

$$\chi_{V^*} = \overline{\chi_V}, \tag{4.3}$$

$$\chi_{S^2 V}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}, \tag{4.4}$$

$$\chi_{\bigwedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2} \tag{4.5}$$

PROOF. Fix $g \in G$, and let $\{\lambda_i\}$ and $\{\mu_j\}$ be the eigenvalues of $\rho_V(g)$ and $\rho_W(g)$, respectively. The matrix $\rho_{V\oplus W}(g)$ has eigenvalues $\{\lambda_i\} \cup \{\mu_j\}$, from which the first formula follows. Similarily, $\rho_{V\otimes W}(g)$ has eigenvalues $\{\lambda_i\mu_j\}$, so its trace equals $\sum_{i,j} \lambda_i\mu_j = (\sum_i \lambda_i)(\sum_j \mu_j) = \chi_V(g) \cdot \chi_W(g)$.

For the third claim, note that since $\rho_V(g)$ is a matrix of finite order, all its eigenvalues have norm one. So the eigenvalues of $\rho_{V^*}(g) = (\rho_V(g)^{-1})^T$ are $\lambda_i^{-1} = \overline{\lambda_i}$, hence the third formula follows.

The final two claims are similar; we prove only the first one and leave the second one as an exercise. The eigenvalues of $\rho_{S^2V}(g)$ are given by $\{\lambda_i\lambda_j \mid i \leq j\}$. So

$$\chi_{S^2V}(g) = \sum_{i\leq j} \lambda_i\lambda_j = \frac{(\sum_i \lambda_i)^2 + \sum_i \lambda_i^2}{2} = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}.$$

□

Exercise 4.5. Consider the symmetric group $\mathfrak{S}_3$. There are three conjugacy classes, with representatives $e, (12), (123)$, so the character of a representation V is determined by the three numbers $\chi_V(e)$, $\chi_V((12))$, $\chi_V((123))$. We have already seen three irreducible representations of $\mathfrak{S}_3$:

- The trivial representation V_{triv}.
- The alternating representation V_{alt} (see Examples 3.5).
- The two-dimensional subrepresentation

  $$V_{\text{stand}} := \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\} \subset \mathbb{C}^3$$

  of the permutation representation $\mathbb{C}^3$, see Exercise 3.27. This is known as the *standard representation.*

For each of these representations, determine the character.

Exercise 4.6. (Exercise 2a on Sheet 2.) Let G be a finite group acting on a finite set X, and let V be the permutation representation. Then $\chi_V(g)$ is the number of elements of X fixed by the action of g.

As mentioned before, one of our main results will be that a representation is uniquely determined by its character. The following exercise can already give some intuition why this might be true.

Exercise 4.7. (Exercise 5 on Sheet 2.) Show that if we know the character χ of a representation $\rho : G \to GL(V)$, then we know the coefficients of the characteristic polynomial of each element $\rho(g)$, and hence the eigenvalues of each $\rho(g)$. Carry this out explicitly for elements $g \in G$ of orders 2, 3, and 4, and for a representation of G on a vector space of dimension 2, 3, or 4.

4.2. The first projection formula. Let V be a representation of a finite group G. Let us try to deduce some information about V, knowing only its character. Recall that $V^G := \{v \in V \mid g \cdot v = v \quad \forall g \in G\}$ the set of vectors fixed by the action of G.

Remark 4.8. If we decompose V into isotypic components, V^G is precisely the isotypic component corresponding to the trivial representation.

Proposition 4.9 (First projection formula)**.** *The linear map*

$$\begin{aligned} \varphi : V &\to V \\ v &\mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v \end{aligned}$$

is a projection of V onto V^G, and moreover G-linear.

Proof. One verifies that (for $h \in G$ and $v \in V$):

(1) $\varphi(h \cdot v) = \varphi(v)$,
(2) $h \cdot \varphi(v) = \varphi(v)$,
(3) $v \in V^G$ implies that $\varphi(v) = v$.

Here is the proof of (2), the other two are left as an exercise:

$$h \cdot \varphi(v) = h \cdot (\frac{1}{|G|} \sum_{g \in G} g \cdot v) = \frac{1}{|G|} \sum_{g \in G} hg \cdot v = \frac{1}{|G|} \sum_{g' \in G} g' \cdot v = \varphi(v).$$

Now (1) and (2) imply that φ is G-linear. Moreover (2) means the image of φ is contained in V^G; together with (3) that means that it is a projection onto V^G. □

Corollary 4.10. *The dimension of V^G can be computed as*

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

Proof. For any projection of a vector space onto a subspace, the dimension of the image is equal to the trace of the projection. So

$$\dim V^G = \operatorname{tr} \varphi = \operatorname{tr}(\frac{1}{|G|} \sum_{g \in G} \rho_V(g)) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). \qquad \square$$

In particular, the multiplicity of the trivial representation in the decomposition of V is uniquely determined by its character. If we can prove the same thing for the multiplicity of *every* irrep inside V, we will have achieved our goal of showing that a representation is uniquely determined by its character.

The trick is to apply Corollary 4.10 to the representation $\operatorname{Hom}(V, W)$, leading to the following powerful result:

Theorem 4.11. *Define a (positive-definite) Hermitian product on $\mathbb{C}_{class}(G)$ by*

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

Then the characters of the irreps of G are orthonormal with respect to this Hermitian product.

Proof. Let V and W be irreps and consider the representation $\mathrm{Hom}(V, W)$. By Exercise 3.19, the fixed point set $\mathrm{Hom}(V, W)^G$ is equal to the space $\mathrm{Hom}_G(V, W)$ of homomorphisms of G-representations. But by Schur's lemma (Exercise 3.25), this space is 1-dimensional if V and W are isomorphic, and 0-dimensional if they are not. On the other hand, the representation $\mathrm{Hom}(V, W) \cong V^* \otimes W$ has character $\chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)} \cdot \chi_W(g)$. So applying Proposition 4.9 gives the formula

$$\frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)} \cdot \chi_W(g) = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \not\cong W. \end{cases}$$

This is exactly what we needed to prove. □

We immediately obtain several interesting consequences from this theorem:

Corollary 4.12. *The number of nonisomorphic irreps of G is at most the number of conjugacy classes in G.*

Proof. Orthonormal vectors are in particular linearly independent, and the dimension of $\mathbb{C}_{\mathrm{class}}(G)$ is equal to the number of conjugacy classes in G. □

In particular, there are only finitely many nonisomorphic irreps. So from now on, whenever we decompose $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ into irreps, we can assume that the sum runs over a complete set of nonisomorphic irreps (with some of the a_i being 0).

Corollary 4.13. *Any representation is uniquely determined by its character.*

Proof. We can write $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$, where the V_i are nonisomorphic irreps. Then $\chi_V = \sum_i a_i \chi_{V_i}$, and since the χ_{V_i} are linearly independent, the a_i are uniquely determined by χ_V. □

Corollary 4.14. *Let $V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ be any representation (where again V_i are nonisomorphic irreps).*

(1) The multiplicity a_i of V_i in V is equal to the product $\langle \chi_V, \chi_{V_i} \rangle$.
(2) We have $\langle \chi_V, \chi_V \rangle = \sum_i a_i^2$.

Proof. Follows immediately from Theorem 4.11. □

Applying Theorem 4.11 to the regular representation we obtain the following:

Corollary 4.15. *The decomposition of R_G into irreps is given by*

$$R_G = \bigoplus_i V_i^{\oplus \dim V_i}$$

where the sum is over **all** *irreps of G.*

Proof. By Exercise 4.6, the character of R_G is given by

$$\chi_{R_G}(g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{else.} \end{cases}$$

So writing $R_G = \bigoplus_i V_i^{\oplus a_i}$ (the sum being over a complete set of nonisomorphic irreps), by Corollary 4.14 we have

$$a_i = \langle \chi_{R_G}, \chi_{V_i} \rangle = \frac{1}{|G|} \cdot |G| \cdot \chi_{V_i}(e) = \dim V_i.$$

□

Since in particular all irreps appear in G, we now have a procedure of constructing all irreps of a given group: just take its regular representation and decompose it into irreducibles.

Remark 4.16. From Corollary 4.15 we see

$$|G| = \dim R_G = \sum_i (\dim V_i)^2. \tag{4.6}$$

Also, evaluating the formula $\chi_{R_G} = \sum_i (\dim V_i) \cdot \chi_{V_i}$ at $g \neq e$ yields

$$\sum_i (\dim V_i) \cdot \chi_{V_i}(g) = 0 \quad \text{for any } g \neq e. \tag{4.7}$$

In the next section we will prove that the number of irreps of G is equal to the number of conjugacy classes. Assuming this for a moment, the formulas (4.6) and (4.7) allow us to compute the character of the final irrep of G, assuming we already know the characters of all other irreps.

4.3. Number of irreps; character tables. We have shown that the number of irreps of G is at most the number of conjugacy classes (Corollary 4.12). To show that this in fact an equality is equivalent to showing that the characters χ_{V_i} don't just form a linearly independent subset, but in fact a basis of $\mathbb{C}_{\text{class}}(G)$. Before we can prove this we need the following lemma, which gives a characterization of class functions:

Lemma 4.17. *Let $\alpha : G \to \mathbb{C}$ be any function. Then α is a class function if and only if for every representation V, the linear map*

$$\begin{aligned} \varphi_{\alpha,V} &: V \to V \\ v &\mapsto \sum_{g \in G} \alpha(g)(g \cdot v) \end{aligned} \tag{4.8}$$

is a morphism of representations.

Proof. Suppose α is a class function, then for any $h \in G$ and $v \in V$:

$$\begin{aligned} \varphi_{\alpha,V}(h \cdot v) &= \sum_g \alpha(g)(gh \cdot v) \\ &= h \cdot \sum_g \alpha(g)(h^{-1}gh \cdot v) \\ &= h \cdot \sum_g \alpha(h^{-1}gh)(h^{-1}gh \cdot v) \qquad \text{since } \alpha \text{ is a class function,} \\ &= h \cdot \sum_{g'} \alpha(g')(g' \cdot v) \\ &= h \cdot \varphi_{\alpha,V}(v), \end{aligned}$$

i.e. $\varphi_{\alpha,V}$ is G-linear.

On the other hand, suppose that for every V, we have that $\varphi_{\alpha,V}$ is G-linear. Then this is true in particular for $V = R_G$ the regular representation. In this case we have $\varphi_{\alpha,V}(e_a) = \sum_{g \in G} \alpha(g)(e_{ga})$. Then G-linearity says that

$$h \cdot \varphi_{\alpha,V}(e_a) = \sum_{g \in G} \alpha(g) e_{hga} = \sum_{g' \in G} \alpha(h^{-1}g'a^{-1}) e_{g'}$$

is equal to

$$\varphi_{\alpha,V}(e_{ha}) = \sum_{g\in G} \alpha(g)e_{gha} = \sum_{g'\in G} \alpha(g'a^{-1}h^{-1})e_{g'}.$$

Comparing coefficients yields that $\alpha(h^{-1}g'a^{-1}) = \alpha(g'a^{-1}h^{-1})$ for every $a, g', h \in G$. This is easily seen to be equivalent to α being a class function (cfr. Exercise 4.3). □

THEOREM 4.18. *The characters of the irreps of a finite group G form an orthonormal basis of $\mathbb{C}_{class}(G)$. In particular, the number of irreps equals the number of conjugacy classes of G.*

PROOF. Linear independence follows from Theorem 4.11. Now take $\alpha \in \mathbb{C}_{\text{class}}(G)$ with $\langle \chi_{V_i}, \alpha\rangle = 0$ for all irreps V_i. We need to show that then $\alpha = 0$.

By Lemma 4.17, φ_{α,V_i} is G-linear for every i, and by Schur's lemma, $\varphi_{\alpha,V_i} = \lambda_i \cdot id_{V_i}$ for some $\lambda_i \in \mathbb{C}$. But

$$\lambda_i = \frac{tr(\varphi_{\alpha,V_i})}{\dim V_i} = \frac{\sum_g \alpha(g)\chi_{V_i}(g)}{\dim V_i} = \frac{|G|\cdot\langle \chi_{V_i^*}, \alpha\rangle}{\dim V_i} = 0.$$

So $\varphi_{\alpha,V_i} = 0$ for every irrep V_i, and hence $\varphi_{\alpha,V} = 0$ for any representation V. Taking $V = R_G$ the regular representation and evaluating at $g = e$ yields

$$0 = \varphi_{\alpha,R_G}(e) = \sum_{g\in G} \alpha(g)e_g.$$

But this means $\alpha(g) = 0$ for every $g \in G$, i.e. $\alpha = 0$. □

Exercise 4.19. Let V_i be an irrep of G, and consider the class function given by

$$\alpha(g) = \frac{\dim V_i}{|G|}\overline{\chi_{V_i}(g)}.$$

Then for any representation V, the map $\varphi_{\alpha,V}$ (4.8) is the projection of V onto the isotypic component corresponding to V_i. Note that the case where V_i is the trivial representation was already done in the proof of Proposition 4.9.

Remark 4.20. The characters of the irreps of a given group G are traditionally presented in the form of a *character table*, whose rows are labeled by the irreps and columns labeled by the conjugacy classes. By Theorem 4.11, the rows of the character table are orthonormal (with respect to the inner product defined there), and by Theorem 4.18 the number of rows equals the number of columns.

Example 4.21. In Exercise 4.5 we computed the characters of three irreps of $\mathfrak{S}_3$. Since $\mathfrak{S}_3$ has only three conjugacy classes, we now know (by Corollary 4.12) these are all the irreps; i.e. we know the character table:

	e	(12)	(123)
V_{triv}	1	1	1
V_{alt}	1	-1	1
V_{stand}	2	0	-1

Since $\#[e] = 1$, $\#[(12)] = 3$, and $\#[(123)] = 2$ (where $[g]$ denotes the conjugacy class of g), the Hermitian inner product $\langle \_, \_\rangle$ of two rows (a_1, a_2, a_3) and (b_1, b_2, b_3) is given by $\frac{1}{6}(a_1b_1 + 3a_2b_2 + 2a_3b_3)$. Note that the rows of the character table are indeed orthonormal with respect to this inner product.

Exercise 4.22. The formulas (4.6) and (4.7) can be generalized to the following orthogonality relations between the columns of the character table: for every $g \in G$ we have

$$\sum_{V_i} \overline{\chi_{V_i}(g)}\chi_{V_i}(g) = \frac{|G|}{c(g)}, \tag{4.9}$$

where $c(g)$ is the number of elements in the conjugacy class of g.

For $g, h \in G$ not conjugate, we have

$$\sum_{V_i} \overline{\chi_{V_i}(g)}\chi_{V_i}(h) = 0. \tag{4.10}$$

Hint. Rescale the elements of the character table so it becomes an orthonormal matrix, and use that the rows of a matrix are orthonormal if and only if the columns are.

5. Associative algebras

We collect here some definitions and basic facts about associative algebras that we will need in the future.

5.1. Algebras and modules.

Definition 5.1. Let k be a field. A *unital associative k-algebra* is a vector space A equipped with a product $\cdot : A \times A \to A$, such that:

(1) $\cdot : A \times A \to A$ is a bilinear map,
(2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in A$ (associativity),
(3) there exists[7] an $e \in A$ such that $e \cdot a = a \cdot e = a$ for all $a \in A$ (unitality).

Note that an associative algebra is in particular a (not necessarily commutative) ring (if we forget the vector space structure).

Definition 5.2. A *left A-module* is a vector space M equipped with a bilinear map $A \times M \to M : (a, m) \mapsto a \cdot m$ such that

(1) $e \cdot m = m$ for all $m \in M$.
(2) $(a \cdot b) \cdot m = a \cdot (b \cdot m)$ for all $a, b \in A$, $m \in M$.

Similarly, a *right A-module* is a vector space M equipped with a bilinear map $M \times A \to M : (m, a) \mapsto m \cdot a$ such that

(1) $m \cdot e = m$ for all $m \in M$.
(2) $m \cdot (a \cdot b) = (m \cdot a) \cdot b$ for all $a, b \in A$, $m \in M$.

You can convince yourself that this agrees with the usual definition of module over a ring.

For the remainder of this subsection we will only work with left modules, but everything works the same for right modules.

Definition 5.3. A *morphism* of left A-modules is a linear map $f : M \to N$ such that $f(am) = af(m)$ for all $a \in A, m \in M$. A *left submodule* $M \subseteq N$ is a vector subspace M of a left A-module M such that $am \in M$ for all $a \in A$ and $m \in M$. A left A-module N is called *irreducible* if the only submodules are 0 and N.

---

[7] Such an e is automatically unique

Remark 5.4. For every algebra A, the vector space A is naturally a left A-module, called the *regular left A-module*. A *left ideal* in A is a left submodule of the regular left A-module.

Definition 5.5. The direct sum $M \oplus N$ of two left A-modules is the vector space $M \oplus N$ with the obvious action $a \cdot (m, n) = (am, an)$. A left A-module M is *indecomposable* if $M = M_1 \oplus M_2$ (with M_1 and M_2 both A-modules) implies $M_1 = 0$ or $M_2 = 0$. Note that irreducible modules are automatically indecomposable (but the other implication does not hold in general).

5.2. Idempotents.

Definition 5.6. An element e of an algebra A is called *idempotent* if $e^2 = e$. Two idempotents e_1 and e_2 are *orthogonal* if $e_1e_2 = e_2e_1 = 0$. An idempotent e is *primitive* if it cannot be written as $e = e_1 + e_2$ with e_1 and e_2 nonzero orthogonal idempotents.

Lemma 5.7. *Let $e \in A$ be an idempotent and consider the left ideal $Ae := \{ae \mid a \in A\}$. Then there is a one-to-one correspondence between:*

(1) Decompositions $e = e_1 + e_2$ of e as a sum of orthogonal idempotents.
(2) Decompositions $Ae = I_1 \oplus I_2$ as a direct sum of left submodules of Ae.

In particular, e is primitive if and only if the left ideal $Ae := \{ae \mid a \in A\}$ is an indecomposable left A-module.

Proof. Suppose $e = e_1 + e_2$ with e_1, e_2 orthogonal idempotents. Then define $I_1 = Ae_1$ and $I_2 = Ae_2$. We verify that $Ae = Ae_1 \oplus Ae_2$:

- If $c \in Ae_1 \cap Ae_2$ then $c = ae_1 = be_2$, and right multiplying with e_1 yields $c = 0$.
- $Ae \subseteq Ae_1 + Ae_2$ since $e = e_1 + e_2$, and $Ae_i \subseteq Ae$ since $e_i = e_ie$.

Conversely, suppose Ae is not indecomposable: $Ae = I_1 \oplus I_2$ with $I_1, I_2 \subset Ae$ nonzero left ideals. There are unique $e_1 \in I_1$ and $e_2 \in I_2$ such that $e = e_1 + e_2$, and $e_1, e_2 \neq 0$. Since $e_1 \in Ae$ we can write $e_1 = ae$. Right multiplying with e gives $e_1e = ae^2 = ae = e_1$. This can be rewritten as $e_1 - e_1^2 = e_1e_2$. Note that the left hand side is in I_1 and the right hand side is in I_2. So since $I_1 \cap I_2 = 0$ we get $e_1 = e_1^2$ and $e_1e_2 = 0$. Similarly we get $e_2 = e_2^2$ and $e_2e_1 = 0$. So e_1 and e_2 are orthogonal idempotents.

We still need to verify that the above procedures are inverse to one another. On the one hand, if we start with a decomposition $e = e_1 + e_2$ then indeed e_1 and e_2 are the unique elements in Ae_1 and Ae_2 such that $e = e_1 + e_2$. On the other hand, if we start with a decomposition $Ae = I_1 \oplus I_2$ and define e_1 and e_2 by the equality $e = e_1 + e_2$, we need to show that $Ae_1 = I_1$ (and similarly for I_2). The inclusion "$\subseteq$" is clear. To show "$\supseteq$", take any element $a \in I_1$. Since $I_1 \subset Ae$, we can write $a = be$. But now $I_1 \ni a = be = be_1 + be_2$; and since the first summand is in I_1 and the second in I_2 we get $be_2 = 0$ and $a = be_1 \in Ae_1$. □

5.3. Bimodules and tensor products.

Definition 5.8. If A and B are two k-algebras, an (A, B)-*bimodule* is a vector space M that is at the same time a left A-module and a right B-module, such that the actions of A and B commute:

$$(a \cdot m) \cdot b = a \cdot (m \cdot b) \quad \text{for every } a \in A, b \in B, m \in M.$$

In other words, we just have a vector space with some ring A acting on the left and some ring B acting on the right, and the actions are compatible. We sometimes will write ${}_AM_B$ to remember which ring is acting from where. A *morphism* of (A,B)-bimodules is a linear map $f : M \to N$ such that $f(amb) = af(m)b$ for all $a \in A, b \in B, m \in M$.

Remark 5.9.
- Left A-modules are the same as (A,k)-bimodules, and right A-modules are the same as (k,A)-bimodules.
- If A is a commutative algebra, left A-modules, right A-modules, and (A,A)-bimodules are the same thing.
- Every algebra A is an (A,A)-bimodule over itself.

Definition 5.10. If now we have three rings A, B, C, an (A,B)-bimodule ${}_AM_B$, and a (B,C)-bimodule ${}_BN_C$, we can define their *tensor product* $M \otimes_B N$ (or more verbose: $({}_AM_B) \otimes_B ({}_BN_C)$), which is an (A,C)-bimodule. It can be constructed as a quotient

$$(M \otimes N)/L, \quad \text{with } L = \mathrm{Span}\{(x \cdot b) \otimes y - x \otimes (b \cdot y) \mid x \in M, y \in N, b \in B\},$$

where $M \otimes N$ is the tensor product of vector spaces from Section 2.2. The actions of A and C are given by

$$a \cdot [x \otimes y] \cdot c = [(ax \otimes yc)].$$

Remark 5.11. (k,k)-bimodules are just vector spaces, and in the case $A = B = C = k$ the tensor product $M \otimes_k N$ defined above is just the tensor product $M \otimes N$ of vector spaces.

Remark 5.12 (*)**.** The tensor product of bimodules satisfies a universal property: the canonical map

$$\phi : M \times N \to M \otimes_B N$$

is the universal[8] bilinear map from $M \times N$ to an (A,C)-bimodule satisfying the following properties (for all $a \in A$, $b \in B$, $c \in C$, $x \in M$, $y \in N$):
- ϕ respects the left A-action: $\phi(ax, y) = a\phi(x, y)$,
- ϕ is B-*balanced*: $\phi(xb, y) = \phi(x, by)$,
- ϕ respects the right C-action: $\phi(x, yc) = \phi(x, y)c$.

6. The group algebra; induced representations

6.1. The group algebra.

Definition 6.1. The *group algebra* $\mathbb{C}G$ of a finite group G is the free vector space $\{e_g \mid g \in G\}$, equipped with a product defined by $e_g \cdot e_h = e_{gh}$.

There is a one-to-one correspondence between G-representations and left $\mathbb{C}G$-modules: if V is a left $\mathbb{C}G$-module it becomes a representation via $g \cdot v := e_g \cdot v$, and if V is a G-representation then it becomes a $\mathbb{C}G$-module via $(\sum_{g\in G} a_g e_g) \cdot v := \sum_{g \in G} a_g(g \cdot v)$. Moreover, the notions "morphism of G-representations" and "morphism of left $\mathbb{C}G$-modules" agree[9]. By definition, irreducible representations

[8]In the exact same sense as Remark 2.5; you can write down the relevant commutative diagram as an exercise.

[9]In category theory language: there is an equivalence of categories between G-representations and left $\mathbb{C}G$-modules

agree with irreducible modules. Maschke's theorem (Theorem 3.26) can be restated as: *every indecomposable $\mathbb{C}G$-module is irreducible.*

Remark 6.2. Here is one more equivalent definition: a representation is a morphism $\mathbb{C}G \to \mathrm{End}(V)$ of unital $\mathbb{C}$-algebras.

Remark 6.3. The regular representation R_G is just the ring $\mathbb{C}G$, viewed as a module over itself. The left ideals in $\mathbb{C}G$ are precisely the subrepresentations of R_G.

Proposition 6.4. *Every left ideal in $\mathbb{C}G$ is of the form $\mathbb{C}G \cdot e$, where e is an idempotent.* [10]

Proof. Whenever $V \subset \mathbb{C}G$ is a subrepresentation, we can apply Theorem 3.26 to write $\mathbb{C}G = V \oplus W$ as $\mathbb{C}G$-modules. In particular we get a G-linear map

$$\pi : \mathbb{C}G \twoheadrightarrow V \subset \mathbb{C}G.$$

Since $\pi(\pi(1) \cdot 1) = \pi(1) \cdot \pi(1)$, the element $e := \pi(1)$ is an idempotent, and $V \subset \mathbb{C}G$ is the ideal generated by e. □

By Lemma 5.7, every primitive idempotent e in $\mathbb{C}G$ gives rise to an irreducible $\mathbb{C}G$-module $\mathbb{C}G \cdot e$. Combining Corollary 4.15 and Proposition 6.4, we see that every irrep arises in this way. In the next chapter, we will construct the irreducible representations of the symmetric group by finding sufficiently many idempotents in the group algebra.

Remark 6.5. Exercise 4.19 says that the isotypic component corresponding to the irrep V_i is the ideal generated by the idempotent

$$\frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)} e_g$$

.

We already know that $R_G \cong \bigoplus_i V_i^{\oplus \dim V_i}$, where the sum is over all irreps V_i. We can refine this statement in terms of the group algebra [11].

Proposition 6.6. * *We have an isomorphism of $\mathbb{C}$-algebras*

$$\mathbb{C}G \cong \bigoplus_i \mathrm{End}(V_i).$$

Proof. For every irrep V_i, we have a morphism $\mathbb{C}G \to \mathrm{End}(V_i)$ of $\mathbb{C}$-algebras, given by the action of $\mathbb{C}G$. So we get a natural morphism $\phi : \mathbb{C}G \to \bigoplus_i \mathrm{End}(V_i)$. Now note that for every representation V, the map $\mathbb{C}G \to \mathrm{End}(V)$ factors over ϕ. Explicitely: if $V = \bigoplus_i V_i^{a_i}$, we get the composition

$$\mathbb{C}G \xrightarrow{\phi} \bigoplus_i \mathrm{End}(V_i) \to \bigoplus_i \mathrm{End}(V_i)^{a_i} \to \mathrm{End}(\bigoplus_i V_i^{a_i}).$$

Taking $V = R_G$ to be the regular representation, we get that $\mathbb{C}G \to \mathrm{End}(R_G)$ factors over ϕ. But $\mathbb{C}G \to \mathrm{End}(R_G)$ is injective, hence ϕ needs to be injective as well. Since $\dim \mathbb{C}G = \sum_i (\dim V_i)^2 = \dim \bigoplus_i \mathrm{End}(V_i)$, it follows that ϕ is also surjective. □

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[10] In other words: $\mathbb{C}G$ is a semisimple algebra.

[11] For the people also attending the course on associative algebras: compare the following statement with the Artin-Wedderburn theorem

6.2. Restriction and induction. For this section, we fix a finite group G and a subgroup H. Given a representation of G, we can restrict it to H:

Definition 6.7. For V a representation of G the *restriction* $\operatorname{Res}_H^G V$ (or simply $\operatorname{Res} V$) is the H-representation with underlying vector space V, and action of H given by restricting the action of G.

Remark 6.8. The character $\chi_{\operatorname{Res} V} : H \to \mathbb{C}$ of $\operatorname{Res}_H^G V$ is just the character of V restricted to H.

We now introduce a construction that goes in the other way: given a representation of the smaller group G, we construct a representation of the bigger group H. This construction is most easily stated using the language of tensor products over the group algebra:

Definition 6.9. For W a $\mathbb{C}H$-module, the *induced representation* $\operatorname{Ind}_H^G W$ is the $\mathbb{C}G$-module defined as the tensor product

$$\operatorname{Ind}_H^G W = \mathbb{C}G \otimes_{\mathbb{C}H} W.$$

(Here we viewed $\mathbb{C}G$ as a $(\mathbb{C}G, \mathbb{C}H)$-bimodule via the multiplication in $\mathbb{C}G$.)

This definition is both short and useful in proofs, but it is not very explicit. We now give a more explicit (but less elegant) description of $\operatorname{Ind}_H^G W$: let G/H denote the set of right cosets (Definition 1.16) and choose for every coset $\sigma \in G/H$ a representative $g_\sigma \in \sigma$. So every element of G can be uniquely written as a product $g_\sigma h$, for some $\sigma \in G/H$ and $h \in H$.

As a vector space, $\operatorname{Ind}_H^G W$ is a direct sum of $|G/H|$ copies of W, labeled by the elements of G/H. We write[12]

$$V = \operatorname{Ind}_H^G W = \bigoplus_{\sigma \in G/H} W^\sigma,$$

where W^σ is just the copy of W corresponding to σ. The group action is defined as follows: if $w \in W^\sigma \subset V$, we define

$$\rho_V(g)(w) := \rho_W(h)(w) \in W^\tau \subset V, \quad \text{where } gg_\sigma = g_\tau h.$$

Exercise 6.10. (Exercise 4 on Sheet 3) Show that the two constructions agree: consider the linear map

$$\mathbb{C}G \otimes_{\mathbb{C}H} W \to \bigoplus_{\sigma \in G/H} W^\sigma$$
$$e_{g_\sigma h} \otimes w \mapsto h \cdot w \in W^\sigma$$

(where $h \cdot w := \rho_W(h)(w)$). Show that this is well-defined and an isomorphism of $\mathbb{C}G$-modules.

Exercise 6.11. (Exercise 2 on Sheet 3) The character of the induced representation $\operatorname{Ind} W = \operatorname{Ind}_H^G W$ can be computed as follows:

$$\chi_{\operatorname{Ind} W}(g) = \sum_{\sigma : g_\sigma^{-1} g g_\sigma \in H} \chi_W(g_\sigma^{-1} g g_\sigma).$$

This is a useful way for constructing characters of a group G, assuming the characters of a subgroup H are known.

[12] This is only a direct sum of vector spaces, not of representations.

Remark 6.12. There is a natural inclusion $W \subseteq \operatorname{Ind} W$, which is compatible with the action of H (so more precisely, it is a morphism $W \subseteq \operatorname{Res}\operatorname{Ind} W$). It can be described either as the map $W \to \mathbb{C}G \otimes_{\mathbb{C}H} W : w \to 1 \otimes w$, or alternatively as the inclusion of the summand W^e into $\bigoplus_{\sigma \in G/H} W^\sigma$, where e denotes the coset containing the identity.

THEOREM 6.13. *Let W be a representation of H and U a representation of G. Then there is a natural isomorphism of vector spaces*

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G W, U) \cong \operatorname{Hom}_H(W, \operatorname{Res}_H^G U)$$

given by sending $\operatorname{Ind}_H^G W \to U$ to the composition $W \to \operatorname{Ind}_H^G W \to U$.

PROOF. This is a special case of the more general statement for algebras: if $B \subset A$ is a subalgebra, U is an A-module and W is a G-module, then there is a natural isomorphism (of vector spaces)

$$\operatorname{Hom}_A(A \otimes_B W, U) \cong \operatorname{Hom}_B(W, U).$$

To show this statement, we note that (by the universal property 5.12) elements of $\operatorname{Hom}_A(A \otimes_B W, U)$ are in bijection with bilinear maps $\phi : A \times W \to U$ satisfying the compatibility conditions $\phi(aa', w) = a\phi(a', w)$ and $\phi(ab, w) = \phi(a, bw)$ (for $a, a' \in A$, $b \in B$, $w \in W$). Given such a bilinear map, we get a B-module homomorphism $\psi : W \to U$ by $\psi(w) = \phi(1, w)$, and conversely given a B-module homomorphism $\psi : W \to U$ we can define $\phi(a, w) = a \cdot \psi(w)$. □

Remark 6.14. For a proof using the explicit construction, see [**FH91**, Proposition 3.17].

Corollary 6.15 (Frobenius reciprocity)**.** *Let W be a representation of H and U a representation of G. Then*

$$\langle \chi_{\operatorname{Ind} W}, \chi_U \rangle_G = \langle \chi_W, \chi_{\operatorname{Res} U} \rangle_H,$$

where $\langle -, - \rangle_G$ and $\langle -, - \rangle_H$ are the Hermitian inner product from Theorem 4.11.

PROOF. By linearity, it suffices to consider the case where U and W are irreducible. Then $\langle \chi_{\operatorname{Ind} W}, \chi_U \rangle_G$ is the multiplicity of U in the direct sum decomposition of $\operatorname{Ind} W$, which by Schur's lemma is equal to the dimension of $\operatorname{Hom}_G(\operatorname{Ind} W, U)$. Similarly, $\langle \chi_W, \chi_{\operatorname{Res} U} \rangle_H$ is the multiplicity of W in the direct sum decomposition of $\operatorname{Res} U$, which is equal to the dimension of $\operatorname{Hom}_H(W, \operatorname{Res} U)$. The result now follows from Theorem 6.13. □

Example 6.16. Let $G = \mathfrak{S}_3$ the permutations of $\{1, 2, 3\}$ and $\mathfrak{S}_2 \cong H \subset G$ be the permutations that fix the element 3. The irreducible representations of $\mathfrak{S}_3$ were described in Exercise 4.5. Since $\mathfrak{S}_2$ is abelian, it has just two irreps and both are one-dimensional: the trivial representation U_{triv} and the alternating representation U_{alt} (which has $\rho_{U_{\text{alt}}}((12)) = -1$). By looking at the characters, we can see that

$$\operatorname{Res}(V_{\text{triv}}) \cong U_{\text{triv}}, \quad \operatorname{Res}(V_{\text{alt}}) \cong U_{\text{alt}}, \quad \operatorname{Res}(V_{\text{stand}}) \cong U_{\text{triv}} \oplus U_{\text{alt}}.$$

So Frobenius reciprocity tells us that

$$\operatorname{Ind}(U_{\text{triv}}) \cong V_{\text{triv}} \oplus V_{\text{stand}}, \quad \operatorname{Ind}(U_{\text{alt}}) \cong V_{\text{alt}} \oplus V_{\text{stand}}.$$

CHAPTER 3

Representation theory of the symmetric group

7. Partitions and Young tableaux

Definition 7.1. A *partition* λ of a natural number d is a tuple $(\lambda_1, \dots, \lambda_k)$ of positive integers such that $d = \lambda_1 + \dots + \lambda_k$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$. Some common notations we will use throughout are

- $\lambda \vdash d$: "λ is a partition of d."
- $|\lambda| = d$: "The sum of the entries of λ is d."
- $\mathrm{len}(\lambda) = k$: "The number of entries in λ is k."

Recall that the symmetric group $\mathfrak{S}_d$ consists of the permutations of the set $[d] = \{1, \dots, d\}$. Elements of $\mathfrak{S}_d$ are often denoted by *cycle decomposition*, which is an expression of the form

$$(a_{1,1}, \dots, a_{1,\lambda_1})(a_{2,1}, \dots, a_{2,\lambda_2}) \cdots (a_{k,1}, \dots, a_{k,\lambda_k}), \tag{7.1}$$

where the occurring numbers $a_{i,j}$ are precisely the numbers $1, \dots, d$ (in particular $d = \sum_{i=1}^{k} \lambda_i$). The cycle decomposition (7.1) corresponds to the permutation that sends $a_{i,j}$ to $a_{i,j+1}$ for $j < \lambda_i$, and sends a_{i,λ_i} to $a_{i,1}$. Clearly every element of $\mathfrak{S}_d$ can be represented by a cycle decomposition, and two cycle decompositions represent the same element of $\mathfrak{S}_d$ if and only if they agree up to permutation of the cycles, and cyclic permutation of the entries in each cycle. To every element σ of $\mathfrak{S}_d$ we associate a partition $\lambda(\sigma) \vdash d$, by picking a cycle decomposition (7.1) for which $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and putting $\lambda(\sigma) := (\lambda_1, \dots, \lambda_k)$.

In Exercise 1.11, we described the conjugacy classes of $\mathfrak{S}_d$. Since this result is fundamental for this entire chapter, I will put the solution of that exercise here as a proposition:

Proposition 7.2.
(1) *If $\sigma, \pi \in \mathfrak{S}_d$ are permutations and σ has cycle decomposition*

$$(a_{1,1}, \dots, a_{1,\lambda_1})(a_{2,1}, \dots, a_{2,\lambda_2}) \cdots (a_{k,1}, \dots, a_{k,\lambda_k}), \tag{7.2}$$

then $\pi \circ \sigma \circ \pi^{-1}$ has cycle decomposition

$$(\pi(a_{1,1}), \dots, \pi(a_{1,\lambda_1}))(\pi(a_{2,1}), \dots, \pi(a_{2,\lambda_2})) \cdots (\pi(a_{k,1}), \dots, \pi(a_{k,\lambda_k})).$$

(2) *The assignment $\sigma \mapsto \lambda(\sigma)$ induces a bijection between the conjugacy classes of $\mathfrak{S}_d$ and the partitions of d.*

Proof. Part (1) is a direct computation: any number in $\{1, \dots, d\}$ is of the from $\pi(a_{i,j})$, and the composition $\pi \circ \sigma \circ \pi^{-1}$ maps that to $\pi(a_{i,j+1})$ by construction. From (1) it immediately follows that conjugate elements give rise to the same partition. But the converse holds as well: if $\lambda(\sigma) = \lambda(\sigma')$ then we can say that σ is given by a cycle decomposition (7.2) and σ' is given by a different cycle decomposition

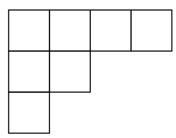
$$(b_{1,1}, \dots, b_{1,\lambda_1})(b_{2,1}, \dots, b_{2,\lambda_2}) \cdots (b_{k,1}, \dots, b_{k,\lambda_k})$$

with same cycle lengths λ_i. But then we can take π to be the permutation that sends $a_{i,j}$ to $b_{i,j}$ for all i, j, and find that $\sigma' = \pi \circ \sigma \circ \pi^{-1}$, i.e. σ and σ' are conjugate. This shows (2). □

Definition 7.3. A *Young diagram* is a finite collection of boxes, arranged in left-aligned rows, with the row lengths in non-increasing order. Young diagrams with d boxes are in bijection with partitions of d: if $\lambda = (\lambda_1, \ldots, \lambda_k)$, then the corresponding Young diagram has λ_i boxes in the i'th row. We will from now on use the concepts "partition" and "Young diagram" interchangeably; in particular the Young diagram associated to a partition λ will be simply denoted by λ. We will use the notation $[\lambda]$ for the set of boxes in the Young diagram λ.

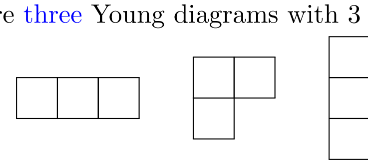
Remark 7.4. For λ a Young diagram, the number of boxes is equal to $|\lambda|$, the number of rows is equal to $\text{len}(\lambda)$, and the number of columns is equal to λ_1.

Example 7.5. Let $\lambda = (4, 2, 1) \vdash 7$, then the corresponding Young diagram is given by



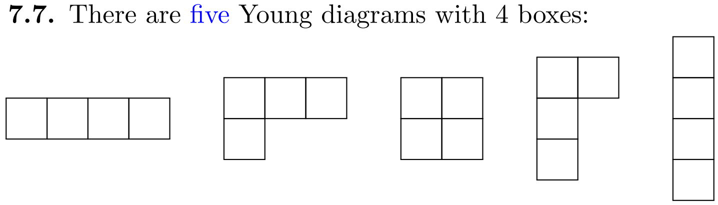
Returning to the symmetric group $\mathfrak{S}_d$, we know that the irreps of $\mathfrak{S}_d$ are in bijection with the conjugacy classes, which are in turn in bijection with Young diagrams with d boxes. In this chapter we will make this explicit, by constructing for any Young diagram an irrep. Moreover, we will see how to read off information about an irrep, like its dimension or even its character, from the corresponding Young diagram.

Example 7.6. There are three Young diagrams with 3 boxes:



These are in bijection with the three irreps of $\mathfrak{S}_3$ listed in Exercise 4.5 and Example 4.21. We will soon see that the left diagram is V_{triv}, the middle one is V_{stand}, and the right one is V_{alt}.

Example 7.7. There are five Young diagrams with 4 boxes:

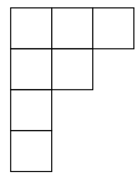


These are in bijection with the five irreps of $\mathfrak{S}_4$ (see the exercise sheets). We will return to this example later.

We close this section with some combinatorial definitions that we will need later:

Definition 7.8. For λ a Young diagram (or partition), the *transposed diagram* (or *conjugate partition*) λ^T is the Young diagram obtained by interchanging the rows and columns: the i'th row of λ^T has as many boxes as the i'th column of λ.

Example 7.9. The transpose of the Young diagram above is the diagram



corresponding to the partition $\lambda^T = (3, 2, 1, 1) \vdash 7$.

Exercise 7.10. An alternative, less visual, definition of conjugate partition is the following: if $\lambda = (\lambda_1, \ldots, \lambda_k)$, then write $\ell = \lambda_1$, and define $\lambda^T := (\mu_1, \ldots, \mu_\ell)$, where $\mu_j = \#\{i \mid \lambda_i \geq j\} = \max\{i \mid \lambda_i \geq j\}$. Convince yourself that this agrees with the definition above.

Definition 7.11. Given a Young diagram $\lambda \vdash d$, a *Young tableau* T on λ is a bijection $[\lambda] \to [d]$ between the boxes of λ and the numbers $1, \ldots, d$. We can visualize such a Young tableau T by numbering the boxes in λ (see Example 7.12). A Young tableau is called *standard* if the entries in each row and column are increasing.

Example 7.12. Here are two Young tableaux on the Young diagram from Example 7.5. The left one is standard, the right one is not.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 4 & 3 & 6 \\ \hline 7 & 5 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}$$

Given a Young diagram $\lambda \vdash d$, there is a natural left action of $\mathfrak{S}_d$ on the set of Young tableaux on λ: the Young tableau $\sigma \cdot T$, viewed as a bijection $[\lambda] \to [d]$, is just the composition $\sigma \circ T$.

Example 7.13. We have

$$(245)(376) \cdot \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 3 & 6 \\ \hline 7 & 5 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array}.$$

Definition 7.14. For every Young diagram λ, we have two maps $\mathrm{row}_\lambda : [\lambda] \to \mathbb{N}$, which maps a box to the index of the corresponding row, and $\mathrm{col}_\lambda : [\lambda] \to \mathbb{N}$, which maps a box to the index of the corresponding column. These will be useful for formally writing down the more visual arguments we will be making later.

Note that in particular for every Young tableau T on λ, we have the map $\mathrm{row}_\lambda \circ T^{-1} : [d] \mapsto \mathbb{N}$ (respectively $\mathrm{col}_\lambda \circ T^{-1} : [d] \mapsto \mathbb{N}$) mapping each number $1, \ldots, d$ to the row (resp. column) it is in.

Example 7.15. If T is the leftmost tableau in Example 7.12, we have $\mathrm{row}_\lambda \circ T^{-1}(6) = 1$ and $\mathrm{col}_\lambda \circ T^{-1}(6) = 3$, since 6 is in the first row and third column.

8. Irreducible representations of the symmetric group

As discussed in Section 6.1, the irreps of $\mathfrak{S}_d$ can be constructed as ideals $\mathbb{C}\mathfrak{S}_d \cdot e$ in the group algebra, where e is a primitive idempotent. So our goal is the following:

> *For every Young diagram $\lambda \vdash d$, construct a primitive idempotent $c_\lambda \in \mathbb{C}\mathfrak{S}_d$, in such a way that the irreps $\mathbb{C}\mathfrak{S}_d \cdot c_\lambda$ are nonisomorphic.*

We will actually do a bit more: for every Young *tableau* we will construct a primitive idempotent $\tilde{c}(T) \in \mathbb{C}\mathfrak{S}_d$, in such a way that two irreps $\mathbb{C}\mathfrak{S}_d \cdot \tilde{c}(T)$ and $\mathbb{C}\mathfrak{S}_d \cdot \tilde{c}(T')$ are isomorphic if and only if the underlying Young diagrams of T and T' are the same. (Different Young tableaux on the same Young diagram will then give rise to different embeddings of the same irrep into $\mathbb{C}\mathfrak{S}_d$.)

Definition 8.1. Let $\lambda \vdash d$ be a Young diagram with d boxes, and let T be a Young tableau on λ.

(1) The subgroup $\mathrm{Row}(T) \subseteq \mathfrak{S}_d$ consists of all permutations which preserve each row of T. More formally:
$$\mathrm{Row}(T) = \{\sigma \in \mathfrak{S}_d \mid \mathrm{row}_\lambda \circ T^{-1} = \mathrm{row}_\lambda \circ T^{-1} \circ \sigma\}.$$
(2) The subgroup $\mathrm{Col}(T) \subseteq \mathfrak{S}_d$ consists of all permutations which preserve each column of T. More formally:
$$\mathrm{Col}(T) = \{\sigma \in \mathfrak{S}_d \mid \mathrm{col}_\lambda \circ T^{-1} = \mathrm{col}_\lambda \circ T^{-1} \circ \sigma\}.$$
(3) We define
$$a(T) = \sum_{\sigma \in \mathrm{Row}(T)} e_\sigma \in \mathbb{C}\mathfrak{S}_d.$$
(4) We define
$$b(T) = \sum_{\sigma \in \mathrm{Col}(T)} \mathrm{sgn}(\sigma) e_\sigma \in \mathbb{C}\mathfrak{S}_d.$$
(5) Finally, we define
$$c(T) := a(T) \cdot b(T) \in \mathbb{C}\mathfrak{S}_d;$$
this element is called the *Young symmetrizer* associated to T.

As promised, here is the main result of this chapter: a construction for all representations of the symmetric group.

THEOREM 8.2. *For any Young tableau T, the $\mathfrak{S}_d$-representation $\mathbb{C}\mathfrak{S}_d \cdot c(T)$ is irreducible. Moreover, two such irreps $\mathbb{C}\mathfrak{S}_d \cdot c(T)$ and $\mathbb{C}\mathfrak{S}_d \cdot c(T')$ are isomorphic if and only if the underling Young diagrams of T and T' are the same.*

So for every partition $\lambda \vdash d$, we can construct an irrep V_λ by picking a Young tableau T on λ and defining[1] $V_\lambda := \mathbb{C}\mathfrak{S}_d \cdot c(T)$. Since we already know from Theorem 4.18 that the number of irreps of $\mathfrak{S}_d$ is equal to the number of partitions of d, Theorem 8.2 tells us that the V_λ, where λ ranges over all partitions of d, form a complete set of pairwise nonisomorphic irreps for $\mathfrak{S}_d$..

Remark 8.3. As we will soon see, the Young symmetrizers $c(T)$ are not quite idempotents; they are idempotents up to scaling. More precisely we will see that $c(T)^2 = n_T c(T)$ for some $n_T \in \mathbb{C} \setminus \{0\}$ (which is equivalent to saying that $\frac{c(T)}{n_T}$ is an idempotent). As the ideals $\mathbb{C}\mathfrak{S}_d \cdot c(T)$ and $\mathbb{C}\mathfrak{S}_d \cdot \frac{c(T)}{n_T}$ are the same this doesn't really make a difference.

[1] This is well-defined as a $\mathfrak{S}_d$-representation, but not as an ideal in $\mathbb{C}\mathfrak{S}_d$ since the latter depends on which tableau T we chose.

Before we start proving Theorem 8.2, let us see it in action for the symmetric group $\mathfrak{S}_3$:

Example 8.4. Consider the Young tableau

$$\mathrm{T}=\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array},$$

then we have $\mathrm{Row}(T) = \mathfrak{S}_3$ and $\mathrm{Col}(T) = \{id\}$, so we have

$$c(T) = a(T) = \sum_{\sigma\in\mathfrak{S}_3} e_\sigma = e_{id} + e_{(123)} + e_{(132)} + e_{(12)} + e_{(13)} + e_{(23)}.$$

Since $c(T)$ is fixed by the action of $\mathfrak{S}_3$, the ideal $\mathbb{C}\mathfrak{S}_3 \cdot c(T)$ is just the one-dimensional vector space $\mathbb{C} \cdot c(T)$; as a representation it is given by the trivial representation:

$$V_{\square\square\square} = V_{\mathrm{triv}}.$$

This example immediately generalizes to any d: if λ is the Young diagram with just one row (that is, the partition (d)), then V_λ is the trivial representation.

Example 8.5. Consider the Young tableau

$$\mathrm{T}=\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array},$$

then we have $\mathrm{Row}(T) = \{id\}$ and $\mathrm{Col}(T) = \mathfrak{S}_3$, so we have

$$c(T) = b(T) = \sum_{\sigma\in\mathfrak{S}_3} \mathrm{sgn}(\sigma) e_\sigma = e_{id} + e_{(123)} + e_{(132)} - e_{(12)} - e_{(13)} - e_{(23)}.$$

Again, the ideal $\mathbb{C}\mathfrak{S}_3 \cdot c(T)$ is just the one-dimensional vector space $\mathbb{C} \cdot c(T)$; but this time as a representation it is given by the alternating representation:

$$V_{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}} = V_{\mathrm{alt}}.$$

This example also generalizes to any d: if λ is the Young diagram with just one column (that is, the partition $(1, \dots, 1)$), then V_λ is the alternating representation.

Example 8.6. Consider the Young tableau

$$\mathrm{T}=\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array},$$

then we have $\mathrm{Row}(T) = \{id, (12)\}$ and $\mathrm{Col}(T) = \{id, (13)\}$, so we have

$$c(T) = (e_{id} + e_{(12)}) \cdot (e_{id} - e_{(13)}) = e_{id} + e_{(12)} - e_{(13)} - e_{(132)}.$$

This time, the ideal $\mathbb{C}\mathfrak{S}_3 \cdot c(T)$ is a two-dimensional vector space spanned by

$$c(T) = e_{id} + e_{(12)} - e_{(13)} - e_{(132)} \text{ and } (13) \cdot c(T) = e_{(13)} + e_{(123)} - e_{id} - e_{(23)}.$$

Since the only 2-dimensional representation of $\mathfrak{S}_3$ is the standard representation, we find

$$V_{\begin{array}{|c|c|}\hline & \\ \hline & \\ \cline{1-1}\end{array}} = V_{\mathrm{standard}}.$$

If we instead pick the Young tableau

$$\mathrm{T'}=\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array},$$

we get

$$c(T') = (e_{id} + e_{(13)}) \cdot (e_{id} - e_{(12)}) = e_{id} + e_{(13)} - e_{(12)} - e_{(123)}.$$

The ideal $\mathbb{C}\mathfrak{S}_3 \cdot c(T')$ is a two-dimensional vector space spanned by

$$c(T') = e_{id} + e_{(13)} - e_{(12)} - e_{(123)} \text{ and } (12) \cdot c(T') = e_{(12)} + e_{(132)} - e_{id} - e_{(23)}.$$

Note that $\mathbb{C}\mathfrak{S}_3 \cdot c(T)$ and $\mathbb{C}\mathfrak{S}_3 \cdot c(T')$ are both isomorphic to $V_{\begin{smallmatrix}\square\square\\ \square\,\,\end{smallmatrix}} = V_{\text{standard}}$, but as ideals (i.e. as subspaces of $\mathbb{C}\mathfrak{S}_3$) they are not the same.

In the example above, note that the span of $\mathbb{C}\mathfrak{S}_3 \cdot c(T)$ and $\mathbb{C}\mathfrak{S}_3 \cdot c(T')$ is the entire isotypic component $V_{\text{standard}}^{\oplus 2}$ in the decomposition of the regular representation $\mathbb{C}\mathfrak{S}_3$. Putting the three previous examples together, we see that a decomposition of $\mathbb{C}\mathfrak{S}_3$ into irreducible subrepresentations is given by

$$\mathbb{C}\mathfrak{S}_3 = \mathbb{C}\mathfrak{S}_3 \cdot c\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) \oplus \mathbb{C}\mathfrak{S}_3 \cdot c\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) \oplus \mathbb{C}\mathfrak{S}_3 \cdot c\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right) \oplus \mathbb{C}\mathfrak{S}_3 \cdot c\left(\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\right).$$

This is a special case of Theorem 8.23, which we will prove later.

8.1. Proof of Theorem 8.2. Throughout this section, we will fix a number d and consider the symmetric group $\mathfrak{S}_d$. We will write $A := \mathbb{C}\mathfrak{S}_d$ for its group algebra. All occurring Young diagrams and Young tableaux have exactly d boxes.

For T a Young diagram, we will write row_T for the map $[d] \to \mathbb{N}$ that maps a number $i \in [d]$ to the row it is in. To connect this with the notation from before: we have $\text{row}_T := \text{row}_\lambda \circ T^{-1}$ and $\text{Row}(T) = \{\sigma \in \mathfrak{S}_d \mid \text{row}_T \circ \sigma = \text{row}_T\}$. Similarly we have $\text{col}_T := \text{col}_\lambda \circ T^{-1}$ and $\text{Col}(T) = \{\sigma \in \mathfrak{S}_d \mid \text{col}_T \circ \sigma = \text{col}_T\}$.

We will start by showing the following:

Proposition 8.7 (First part of Theorem 8.2)**.** *If T and T' are Young tableaux on the same Young diagram λ, then the representations $A \cdot c(T)$ and $A \cdot c(T')$ are isomorphic.*

Proof. First we show that for any $g \in \mathfrak{S}_d$

$$\text{Row}(g \cdot T) = g \cdot \text{Row}(T) \cdot g^{-1} \quad \text{and} \quad \text{Col}(g \cdot T) = g \cdot \text{Col}(T) \cdot g^{-1}.$$

Indeed, note that $\text{row}_{g \cdot T} = \text{row}_T \circ g^{-1}$, so that

$$\begin{aligned}\sigma \in \text{Row}(g \cdot T) &\iff \text{row}_T \circ g^{-1} = \text{row}_T \circ g^{-1} \circ \sigma \\ &\iff \text{row}_T = \text{row}_T \circ g^{-1} \circ \sigma \circ g \\ &\iff g^{-1} \circ \sigma \circ g \in \text{Row}(T) \\ &\iff \sigma \in g \cdot \text{Row}(T) \cdot g^{-1}\end{aligned}$$

and similarly for Col.

Next, we show that $a(g \cdot T) = g \cdot a(T) \cdot g^{-1}$. Indeed:

$$\begin{aligned}a(g \cdot T) &= \sum_{\sigma \in \text{Row}(g \cdot T)} e_\sigma \\ &= \sum_{\sigma' \in \text{Row}(T)} e_{g\sigma' g^{-1}} \\ &= g \cdot a(T) \cdot g^{-1}.\end{aligned}$$

By a similar computation we find that $b(g \cdot T) = g \cdot b(T) \cdot g^{-1}$, and since $c(T) = a(T)b(T)$ we also obtain

$$c(g \cdot T) = g \cdot c(T) \cdot g^{-1}.$$

Finally, let T and T' be Young tableaux on the same diagram. Then there exists a permutation $g \in \mathfrak{S}_d$ such that $T' = g \cdot T$. Then $c(T') = g \cdot c(T) \cdot g^{-1}$, and hence $A \cdot c(T') = A \cdot c(T) \cdot e_{g^{-1}}$. So the map $A \to A : x \mapsto x \cdot e_{g^{-1}}$ restricts to an isomorphism $A \cdot c(T) \to A \cdot c(T')$ (with inverse given by $x \mapsto x \cdot e_g$). □

Observation 8.8. *The Young symmetrizer $c(T)$ has the following property: for every $\sigma \in \mathrm{Row}(T)$ and $\tau \in \mathrm{Col}(T)$, it holds that $\sigma \cdot c(T) \cdot \tau = \mathrm{sgn}(\tau) \cdot c(T)$.*

PROOF. We have

$$\begin{aligned}\sigma \cdot c(T) \cdot \tau =& \sigma \cdot a(T) \cdot b(T) \cdot \tau \\ =& e_\sigma \cdot \left(\sum_{\sigma' \in \mathrm{Row}(T)} e_{\sigma'} \right) \cdot \left(\sum_{\tau' \in \mathrm{Col}(T)} \mathrm{sgn}(\tau') e_{\tau'} \right) \cdot e_\tau \\ =& \mathrm{sgn}(\tau) \cdot \left(\sum_{\sigma'' \in \mathrm{Row}(T)} e_{\sigma''} \right) \cdot \left(\sum_{\tau'' \in \mathrm{Col}(T)} \mathrm{sgn}(\tau'') e_{\tau''} \right) \\ =& \mathrm{sgn}(\tau) \cdot c(T).\end{aligned}$$

□

Our next goal (which we will reach in Lemma 8.15) will be to show that $c(T)$ is the only element of A satisfying the above property. The main tool for this will be the combinatorial Lemma 8.12, which is a converse to the following observation:

Observation 8.9. *Let T be a Young tableau, and pick $\sigma \in \mathrm{Row}(T)$, $\tau \in \mathrm{Col}(T)$. Write $T' = \sigma \cdot \tau \cdot T$, and let $i \neq j \in [d]$ be in the same row of T. Then in T', the numbers i and j are not in the same column.*

PROOF. We have that $\mathrm{row}_T(i) = \mathrm{row}_T(j)$. Since $\sigma^{-1} \in \mathrm{Row}(T)$, we get

$$\mathrm{row}_T(\sigma^{-1}(i)) = \mathrm{row}_T(\sigma^{-1}(j)). \tag{8.1}$$

Now suppose by contradiction that $\mathrm{col}_{T'}(i) = \mathrm{col}_{T'}(j)$. Then

$$\mathrm{col}_T(\tau^{-1}(\sigma^{-1}(i))) = \mathrm{col}_T(\tau^{-1}(\sigma^{-1}(j))).$$

Since $\tau^{-1} \in \mathrm{Col}(T)$, this implies

$$\mathrm{col}_T(\sigma^{-1}(i)) = \mathrm{col}_T(\sigma^{-1}(j)). \tag{8.2}$$

But (8.1) and (8.2) imply that $\sigma^{-1}(i) = \sigma^{-1}(j)$, so $i = j$, a contradiction. □

Definition 8.10. We order the partitions lexicographically:

$$\lambda < \mu \iff \exists i \in \mathbb{N} \text{ with } \lambda_j = \mu_j \text{ for } j < i, \text{ and } \lambda_i < \mu_i.$$

This defines a total ordering on the set of partitions of d. In words: $\lambda < \mu$ if for the first index i in which λ and μ differ, we have $\lambda_i < \mu_i$.

Example 8.11. For the partitions of $d = 5$, we have

$$(1, 1, 1, 1, 1) < (2, 1, 1, 1) < (2, 2, 1) < (3, 1, 1) < (3, 2) < (4, 1) < (5)$$

Now we are ready to state our main combinatorial lemma.

Lemma 8.12. *Let T and T' be Young tableaux on the diagrams λ and μ respectively, with $\lambda \geq \mu$. Suppose that for every $i \neq j \in [d]$ in the same row of T, we have that i and j are not in the same column of T'. Then $\lambda = \mu$, and there exist $\sigma \in \mathrm{Row}(T)$ and $\tau \in \mathrm{Col}(T)$ such that $T' = \sigma \cdot \tau \cdot T$.*

PROOF. If $\lambda_1 > \mu_1$ then μ has fewer columns than λ, so two of the numbers in the first row of T need to be in the same column of T', a contradiction. So $\lambda_1 = \mu_1$, and we find some $\tau_1 \in \mathrm{Col}(T')$ such that the first rows of T and $\tau_1 \cdot T'$ have the same elements.

Now consider T and $\tau_1 \cdot T'$ and ignore their first rows. Repeating the above argument, we find that $\lambda_2 = \mu_2$, and we can find $\tau_2 \in \mathrm{Col}(T')$ such that in both the first and the second rows of T and $\tau_2 \cdot \tau_1 \cdot T'$ have the same elements.

Eventually we find that $\lambda = \mu$, and we can find $\tau' \in \mathrm{Col}(T')$ such that T and $\tau' \cdot T'$ have the same numbers in each row. This means there exists a $\sigma \in \mathrm{Row}(T)$ such that $\sigma \cdot T = \tau' \cdot T'$. Now

$$\tau' \in \mathrm{Col}(T') = \mathrm{Col}(\tau' \cdot T') = \mathrm{Col}(\sigma \cdot T) = \sigma \cdot \mathrm{Col}(T) \cdot \sigma^{-1}.$$

(For the first equality: if $\tau' \in \mathrm{Col}(T')$ then $\mathrm{Col}(T') = \mathrm{Col}(\tau' \cdot T')$).

So if we define $\tau := \sigma^{-1} \cdot \tau'^{-1} \cdot \sigma$, then $\tau \in \mathrm{Col}(T)$, and

$$\sigma \cdot \tau \cdot T = \tau'^{-1} \cdot \sigma \cdot T = T'.$$

□

We need to understand the Young symmetrizers $c(T)$ a bit better.

Lemma 8.13. *Let T be a Young tableau.*

(1) $\mathrm{Row}(T) \cap \mathrm{Col}(T) = \{id\}$.
(2) Every $g \in \mathfrak{S}_d$ can be written in at most one way as a product $\sigma \cdot \tau$ with $\sigma \in \mathrm{Row}(T)$ and $\tau \in \mathrm{Col}(T)$.

PROOF. Part (1) is immediate: if $g \in \mathrm{Row}(T) \cap \mathrm{Col}(T)$ means that g needs to map every $i \in [d]$ to a number that is both in the same row and the same column as i, but the only such number is i. Part (2) follows from (1): if $\sigma\tau = \sigma'\tau'$ then

$$\sigma'^{-1}\sigma = \tau'\tau^{-1} \in \mathrm{Row}(T) \cap \mathrm{Col}(T) = \{id\},$$

so $\sigma = \sigma'$ and $\tau = \tau'$. □

The above lemma gives us an alternative way of thinking about $c(T)$:

Definition 8.14. Write

$$RC(T) := \{g \in \mathfrak{S}_d \mid g = \sigma \cdot \tau \text{ for some } \sigma \in \mathrm{Row}(T), \tau \in \mathrm{Col}(T)\}.$$

For $g = \sigma \cdot \tau \in RC(T)$, we will write $\mathrm{sgn}_T(g) := \mathrm{sgn}(\tau)$, which is well-defined by Lemma 8.13 (2).

Using this definition, we can write

$$c(T) = \sum_{g \in RC(T)} \mathrm{sgn}_T(g) e_g.$$

In particular, note that the coefficient of e_{id} is equal to $\mathrm{sgn}_T(id) = \mathrm{sgn}(id) = 1$.

We can now prove the converse of Observation 8.8:

Lemma 8.15. *Let T be a Young tableau and $x \in A$. Suppose that*

$$\sigma x \tau = \operatorname{sgn}(\tau) x \text{ for all } \sigma \in \operatorname{Row}(T), \tau \in \operatorname{Col}(T). \tag{8.3}$$

Then $x = \alpha c(T)$ for some $\alpha \in \mathbb{C}$.

PROOF. We can write $x = \sum_{g \in \mathfrak{S}_d} a_g e_g$.
Step 1. For $g \notin RC(T)$ the coefficient a_g is equal to 0.

If we write $T' = g \cdot T$, then by Lemma 8.12, we find $i \neq j$ in the same row of T and in the same column of T'. So the transposition (ij) is both in $\operatorname{Row}(T)$ and in $\operatorname{Col}(T') = g \cdot \operatorname{Col}(T) \cdot g^{-1}$. So we can apply our assumption (8.3) to $\sigma = (ij)$ and $\tau = g^{-1}(ij)g$ (note that $\operatorname{sgn}(\tau) = -1$):

$$\sum_{h \in \mathfrak{S}_d} a_h e_{(ij)hg^{-1}(ij)g} = - \sum_{g' \in \mathfrak{S}_d} a_{g'} e_{g'}.$$

Comparing coefficients of e_g yields that $a_g = -a_g$, so $a_g = 0$.
Step 2. Now if we pick $g = \sigma\tau \in RC(T)$, and look at the coefficient of e_g in our equality $\sigma x \tau = \operatorname{sgn}(\tau) x$, we find $a_{id} = \operatorname{sgn}(\tau) a_{\sigma\tau} = \operatorname{sgn}_T(g) a_g$. Hence

$$a = \sum_{g \in RC(T)} a_g e_g = \sum_{g \in RC(T)} \operatorname{sgn}_T(g) a_{id} e_g = a_{id} c(T).$$

□

Corollary 8.16. *For every $y \in A$ we have $c(T) y c(T) = \alpha c(T)$ for some $\alpha \in \mathbb{C}$.*

PROOF. Note that, by a similar computation as Observation 8.8, the element $x := c(T) y c(T)$ satisfies the condition (8.3). So the statement follows from Lemma 8.15. □

Taking $y = 1$ in Corollary 8.16 yields that $c(T)^2 = n_T c(T)$ for some $n_T \in \mathbb{C}$. In the following lemma we compute n_T.

Lemma 8.17. *We have $c(T)^2 = n_T c(T)$, where $n_T = \frac{d!}{\dim(A \cdot c(T))}$.*

PROOF. Write $c = c(T)$. We know that $c^2 = \alpha c$ for some $\alpha \in \mathbb{C}$. We consider the linear map $f : A \to A$ given by right multiplication by c and compute its trace in two ways:

- Since $\operatorname{im}(f) = Ac$, we have $\operatorname{tr}(f) = \operatorname{tr}(f\big|_{Ac})$ (take a basis of Ac, extend to A, and look at the matrix of f). But since $(ac)c = n_T ac$ for each $a \in A$, we get that $f\big|_{Ac}$ is just multiplication by α. So $\operatorname{tr}(f) = \alpha \cdot \dim(Ac)$.
- With respect to the standard basis $\{e_\sigma \mid \sigma \in \mathfrak{S}_d\}$ of A, the matrix of f has (σ, τ)'th entry given by the coefficient of e_σ in $\tau \cdot c(T)$. In particular, we can verify that the diagonal entries are equal to 1, therefore $\operatorname{tr}(f) = d!$.

□

In particular we see that $n_T \neq 0$. We will write $\tilde{c}(T) := \frac{c(T)}{n_T}$ for the normalized Young symmetrizer. By the previous lemma, we have $\tilde{c}(T)^2 = \tilde{c}(T)$, i.e. $\tilde{c}(T)$ is an idempotent in the group algebra A.

Proposition 8.18 (Second part of Theorem 8.2)**.** *The $\mathfrak{S}_d$-representation $A \cdot \tilde{c}(T)$ is irreducible.*

PROOF. In the light of Lemma 5.7, we need to show that $\tilde{c}(T)$ is a *primitive* idempotent. So suppose $\tilde{c}(T) = e_1 + e_2$ for e_1, e_2 orthogonal idempotents. Then Corollary 8.16 tells us that $\tilde{c}(T)e_1\tilde{c}(T) = \alpha\tilde{c}(T)$ for some $\alpha \in \mathbb{C}$. But writing this out we get $e_1 = \alpha \cdot (e_1 + e_2)$. Multiplying both sides by e_2 we get $0 = \alpha e_2$. So either $e_1 = 0$ or $e_2 = 0$. □

Lemma 8.19. *If $\lambda > \mu$ and T, T' are Young tableaux on λ, μ respectively, then $c(T')Ac(T) = 0$.*

PROOF. Pick any $g \in \mathfrak{S}_d$. Since $\lambda > \mu$, by Lemma 8.12 there are i, j in the same row of $g \cdot T$ and column of T'. Write $\tau = (ij) \in \mathrm{Row}(g \cdot T) \cap \mathrm{Col}(T')$. Then

$$\begin{aligned} c(T') \cdot e_g \cdot c(T) &= c(T') \cdot c(g \cdot T) \cdot e_g = c(T') \cdot e_\tau \cdot e_\tau \cdot c(g \cdot T) \cdot e_g \\ &= -c(T') \cdot c(g \cdot T) \cdot e_g = -c(T') \cdot e_g \cdot c(T). \end{aligned}$$

So $c(T') \cdot e_g \cdot c(T) = 0$. Since g was arbitrary we get $c(T')Ac(T) = 0$. □

Exercise 8.20. (Not necessary for proof.) Also if $\lambda < \mu$ we have $c(T')Ac(T) = 0$.

Proposition 8.21 (Third part of Theorem 8.2)**.** *If T and T' are Young tableaux on distinct Young diagrams λ, μ, then $Ac(T)$ and $Ac(T')$ are not isomorphic.*

PROOF. We can assume $\lambda > \mu$. If there is an isomorphism $f : Ac(T') \to Ac(T)$ of $\mathfrak{S}_d$-representations, then $f(\tilde{c}(T')) = x\tilde{c}(T)$ for some $x \in A$, but then

$$f(\tilde{c}(T')) = f(\tilde{c}(T')^2) = \tilde{c}(T')f(\tilde{c}(T')) = \tilde{c}(T')x\tilde{c}(T) = 0,$$

a contradiction with f being an isomorphism. □

8.2. Standard tableaux. The goal of this section is to prove the following statement, which gives us one way of reading off the dimension of an irrep V_λ from the Young diagram λ.

THEOREM 8.22. *The dimension of V_λ is equal to the number of standard Young tableaux on λ.*

This is a corollary of the following:

THEOREM 8.23. *The group ring $\mathbb{C}\mathfrak{S}_d$ decomposes as a direct sum of left ideals*

$$\mathbb{C}\mathfrak{S}_d = \bigoplus_T \mathbb{C}\mathfrak{S}_d \cdot c(T) \tag{8.4}$$

where the sum is over all standard Young tableaux T of size d.

PROOF OF THEOREM 8.22 ASSUMING THEOREM 8.23. Fix a partition $\lambda \vdash d$. By Corollary 4.15, the multiplicity of V_λ in the regular representation is equal to $\dim V_\lambda$. But looking at (8.4), and recalling that $\mathbb{C}\mathfrak{S}_d \cdot c(T)$ is an irrep which is isomorphic to V_λ if and only if λ is the underlying tableau of T, this multiplicity is precisely the number of standard Young tableaux on λ. □

There are two main ingredients going into the proof of Theorem 8.23. The first (Lemma 8.26) is that for T and T' standard Young tableaux, the product of the Young symmetrizers $c(T)$ and $c(T')$ is zero. The second (Proposition 8.27) is a purely combinatorial statement about the number of standard Young tableaux.

Definition 8.24. Let T and T' be standard Young tableaux with d boxes, and let λ and μ be their underlying diagrams.

- If $\lambda \neq \mu$ we say $T > T'$ if and only if $\lambda > \mu$ (Definition 8.10).
- If $\lambda = \mu$ we say $T > T'$ if when we read T and T' lexicographically (i.e. like a book), $T(b) < T'(b)$ for $b \in [\lambda]$ the first box in which they differ.

  If you find the above too informal: $T > T'$ if there exists a $b \in [\lambda]$ such that
  - $T(b') = T'(b')$ whenever $\text{row}_\lambda(b') < \text{row}_\lambda(b)$.
  - $T(b') = T'(b')$ whenever $\text{row}_\lambda(b') = \text{row}_\lambda(b)$ and $\text{col}_\lambda(b') < \text{col}_\lambda(b)$.
  - $T(b) < T'(b)$.

This defines a total order on all standard Young tableaux with d boxes.

Example 8.25. Here are the standard Young tableaux on $\lambda = (3, 2)$, ordered according to the above definition:

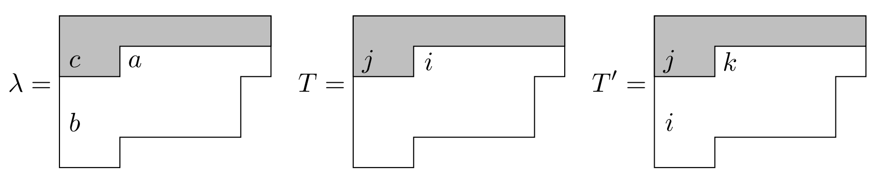
$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} > \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Lemma 8.26. *If $T > T'$ are standard with d boxes, then $c(T') \cdot c(T) = 0$.*

Proof. If T and T' have distinct underlying diagram, then this follows from Lemma 8.19. So assume they have the same underlying diagram λ.
Step 1. Find $i \neq j$ in same row of T and same column of T'.

Let $a \in [\lambda]$ be the lexicographically first box where T and T' differ. We write $i = T(a)$ and $k = T'(a)$ (so $i < k$). Let $b = T'^{-1}(i)$ be the box in T' containing i. Note that b can not come lexicographically before a (since T and T' agree in these positions), and can also not be down and to the right of a, since that would contradict T' being a standard tableau. So b needs to be stricty down and strictly to the left of a. Now we can let $c \in [\lambda]$ be the box in the same row of a and the same column of b, and let $j = T(c) = T'(c)$.

Here is a pictorial representation of Step 1. The shaded area is the part where T and T' agree:



Step 2. Proceed as in Lemma 8.19: let τ be the transposition (ij), by construction $\tau \in \text{Row}(T) \cap \text{Col}(T')$. But then

$$c(T') \cdot c(T) = c(T') \cdot e_\tau \cdot e_\tau \cdot c(T) = -c(T') \cdot c(T),$$

hence $c(T') \cdot c(T) = 0$. □

Proposition 8.27. *Let f_λ denote the number of standard Young tableaux on λ. Then for any $d \in \mathbb{N}$ we have $\sum_{\lambda \vdash d} f_\lambda^2 = d!$.*

We will use two combinatorial lemma's in the proof of Proposition 8.27. The first one is rather straightforward, the second one is much harder. We will write $\mu \to \lambda$ if the Young diagram λ is obtained from μ by either adding a single box to one of the rows, or by adding a new row of length 1. In the language of partitions: $\mu \to \lambda$ means that either

$$\lambda = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_k)$$

for some $i \in \{1, \ldots, k\}$, or that

$$\lambda = (\mu_1, \ldots, \mu_k, 1).$$

Note that in the first case we need to have $\mu_i < \mu_{i+1}$.

Lemma 8.28. *For $\lambda \vdash d$, there is a bijection*

$$\{\textit{standard Young tableaux on } \lambda\} \leftrightarrow \bigcup_{\mu \to \lambda} \{\textit{standard Young tableaux on } \mu\}.$$

Proof. Given a standard tableau T on λ, the box labeled d is always the last box in its row and column, and if we remove it we obtain a standard tableau on some $\mu \to \lambda$. Conversely, if we start with a $\mu \to \lambda$ and a standard tableau T' on it, we can add a box labeled d on the correct location to get a standard tableau on λ. □

Lemma 8.29. *For $\mu \vdash d - 1$, there is a bijection*

$$\{\textit{standard Young tableaux on } \mu\} \times \{1, \ldots d\} \leftrightarrow \bigcup_{\lambda \leftarrow \mu} \{\textit{standard Young tableaux on } \lambda\}.$$

Proof sketch*. The bijection is given by the *row-bumping algorithm*. Given a standard tableau T on μ, and a number $j \in \{1, \ldots, d\}$, we first increase the label of every box labeled $j, \ldots, d$ by one. Next we add a box labeled j to this tableau, by performing the following steps:

(1) We first try to add a box labeled j to the first row:
  (a) If j is greater than every entry in the first row, we just add j to the end of the first row and are done with the algorithm.
  (b) If not, find the leftmost box in the first row with entry greater than j; call this entry j_1; relabel the box by j, and continue to step (2).
(2) Next we try to add a box labeled j_1 to the second row:
  (a) If j_1 is greater than every entry in the second row, we add j_1 to the second row and are done with the algorithm.
  (b) If not, find the leftmost box in the second row with entry greater than j_1; call this entry j_2, relabel the box by j_1, and go to step (3).
(3) Repeat the above step for the third, fourth, ..., row, until either we are in case (a). If we reach the final row without ever being in case (a), we add an extra row with a single box labeled j_k.

The output of this algorithm is always[2] a standard tableau on some $\lambda \leftarrow \mu$. So we get at least a map from left to right. To show it is a bijection, one argues that given a standard tableaux on some $\lambda \leftarrow \mu$, we can remove the box that is in λ but not in μ by performing the steps above in reverse. □

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[2]Both this statement and the next one require some verification, which is why this is only a proof sketch and not a proof.

Proof of Proposition 8.27. By the above lemmas, we have $f_\lambda = \sum_{\mu\to\lambda} f_\mu$ and $d \cdot f_\mu = \sum_{\lambda\leftarrow\mu} f_\lambda$. So

$$\begin{aligned}
\sum_{\lambda\vdash d} f_\lambda^2 &= \sum_{\lambda\vdash d} f_\lambda \cdot \left(\sum_{\mu\to\lambda} f_\mu\right) \\
&= \sum_{\lambda\vdash d}\sum_{\mu\to\lambda} f_\lambda \cdot f_\mu \\
&= \sum_{\mu\vdash d-1}\sum_{\lambda\leftarrow\mu} f_\lambda \cdot f_\mu \\
&= \sum_{\mu\vdash d-1} f_\mu \cdot \left(\sum_{\lambda\leftarrow\mu} f_\lambda\right) \\
&= d \cdot \sum_{\mu\vdash d-1} f_\mu^2 \\
&= d \cdot (d-1)! = d! \qquad\qquad \text{(induction on } d.\text{)}
\end{aligned}$$

□

Proof of Theorem 8.23. We first show that the sum is direct. So assume by contradiction we can find elements $q(T) \in \mathbb{C}\mathfrak{S}_d$ such that

$$\sum_T q(T) \cdot \tilde{c}(T) = 0, \tag{8.5}$$

where as usual the sum is over all standard tableaux with d boxes. Let T_0 be the maximal (with respect to the order defined in Definition 8.24) standard tableau for which $q(T) \cdot \tilde{c}(T_0) \neq 0$. Then right multiplying (8.5) with $\tilde{c}(T_0)$ and using Lemma 8.26 we find

$$0 = \sum_{T\le T_0} q(T)\cdot \tilde{c}(T) \cdot \tilde{c}(T_0) = q(T_0)\cdot \tilde{c}(T_0)^2 = q(T_0)\cdot\tilde{c}(T_0),$$

a contradiction.

We have now shown that

$$\mathbb{C}\mathfrak{S}_d \supseteq \bigoplus_T \mathbb{C}\mathfrak{S}_d \cdot c(T), \tag{8.6}$$

and want to show this is actually an equality. Let us write D_λ for the dimension of the irrep V_λ. As representations, the left hand side of (8.6) is isomorphic to

$$\bigoplus_{\lambda\vdash d} V_\lambda^{D_\lambda}$$

by Corollary 4.15 (in particular we have $d! = \dim \mathfrak{S}_d = \sum D_\lambda^2$), and the right hand side is isomorphic to

$$\bigoplus_{\lambda\vdash d} V_\lambda^{f_\lambda}.$$

So (8.6) implies that we have $f_\lambda \le D_\lambda$ for every λ. But now Proposition 8.27 implies that we need to have an equality $\sum f_\lambda^2 = d! = \sum D_\lambda^2$. This can only hold if $f_\lambda = D_\lambda$ for each $\lambda \vdash d$, finishing the proof. □

8.3. The hook length formula and Frobenius character formula. To close the chapter on $\mathfrak{S}_d$, we state without proof two more important results about the representation theory of the symmetric group. The first one is an alternative formula to compute the dimension of an irreducible representation from its Young diagram.

Definition 8.30. Let $\lambda \vdash d$ be a Young diagram. The *hook length* $h_\lambda(b)$ of a box $b \in [\lambda]$ is equal to the number of boxes in λ that are either directly below or directly to the right of b (where the box b itself is counted once).

Example 8.31. Below is the Young diagram corresponding to the partition $(4, 2, 1)$, where in every box b we wrote its hook length $h_\lambda(b)$:

6	4	2	1
3	1		
1			

Theorem 8.32 (Hook length formula). *Let $\lambda \vdash d$ be a Young diagram and V_λ be the corresponding irreducible $\mathfrak{S}_d$-representation. Then*

$$\dim V_\lambda = \frac{d!}{\prod_{b\in[\lambda]} h_\lambda(b)}.$$

Example 8.33. For $\lambda \vdash 7$ the Young diagram from Example 8.31, we have

$$\dim V_\lambda = \frac{7!}{6\cdot 4\cdot 2\cdot 1\cdot 3\cdot 1\cdot 1} = 35.$$

Remark 8.34. Here are two possible ways of proving Theorem 8.32:

(1) By Theorem 8.22, it suffices to show that the hook length formula counts the number of standard tableaux. This is a purely combinatorial statement for which several direct proofs are known, but none of them are easy.
(2) Theorem 8.32 can also be deduced as a corollary of Frobenius' character formula below.

Theorem 8.35. *Let V_λ be an irrep of $\mathfrak{S}_d$, where $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash d$. Let $g \in \mathfrak{S}_d$, and write $\mu = (\mu_1, \ldots, \mu_m) \vdash d$ for its cycle shape. Then we can compute the character $\chi_{V_\lambda}(g)$ as the coefficient of the monomial*

$$x_1^{\lambda_1+k-1} x_2^{\lambda_2+k-2} \cdots x_k^{\lambda_k}$$

in the polynomial

$$\prod_{1\le i<j\le k} (x_i - x_j) \prod_{i=1}^{m} x_1^{\mu_i} + \cdots + x_k^{\mu_i}.$$

A proof of this theorem can be found in [**FH91**, Lecture 4].

Example 8.36. Let $\lambda = (3, 2) \vdash 5$ and consider the $\mathfrak{S}_5$-representation V_λ. Take $g = (12) \in \mathfrak{S}_5$ a transposition, then $\mu = (2, 1, 1, 1)$. Then $\chi_{V_\lambda}(g)$ is the coefficient of the monomial $x_1^4 x_2^2$ in the polynomial

$$(x_1 - x_2)(x_1^2 + x_2^2)(x_1 + x_2)^3 = x_1^6 + 2x_1^5 x_2 + x_1^4 x_2^2 - x_1^2 x_2^4 - 2x_1 x_2^5 - x_2^6,$$

so we find $\chi_{V_\lambda}(g) = 1$.

CHAPTER 4

Representation theory of the general linear group

9. Polynomial and rational representations of $GL(V)$

In this chapter, we will fix an n-dimensional $\mathbb{C}$-vector space V, and study finite-dimensional representations of the infinite group $GL(V)$. Such a representation is given by a group homomorphism $\rho : GL(V) \to GL(W)$, where W is another finite-dimensional $\mathbb{C}$-vector space.

Let's write $\dim W = m$. If we fix bases of V and W we get identifications $GL(V) \cong GL(n, \mathbb{C})$ and $GL(W) \cong GL(m, \mathbb{C})$, and our representation is given by a map of matrices

$$GL(n, \mathbb{C}) \cong GL(V) \xrightarrow{\rho} GL(W) \cong GL(m, \mathbb{C}). \tag{9.1}$$

It is hard to make interesting statements about arbitrary representations of $GL(V)$. However, $GL(V)$ is not just a group, it comes with a geometric structure[1], and we want to restrict ourselves to representations that preserve that geometric structure.

Definition 9.1. A representation $\rho : GL(V) \to GL(W)$ is *polynomial* (respectively *rational*) if there exist bases of V and W and polynomials (resp. rational functions) $P_{k\ell}$ (for $1 \leq k, \ell \leq m$) in n^2 variables such that the map (9.1) is given by

$$(a_{ij})_{1 \leq i,j \leq n} \mapsto (P_{k\ell}(a_{ij}))_{1 \leq k, \ell \leq m}.$$

Exercise 9.2. In the above definition, we can replace "there exist bases of V and W and polynomials (resp. rational functions) $P_{k\ell}$" by "for all bases of V and W, there exist polynomials (resp. rational functions) $P_{k\ell}$".

Examples 9.3.
- The identity map $GL(V) \to GL(V)$ is an n-dimensional representation, known as the *standard representation*. It is clearly a polynomial representation.
- The determinant map $GL(V) \to GL(1, \mathbb{C}) \cong \mathbb{C}^*$ is a one-dimensional representation of $GL(V)$. We will denote it by Det.
- More generally, for every $a \in \mathbb{Z}$, the map

$$\begin{aligned} GL(V) &\to \mathbb{C}^* \\ A &\mapsto (\det A)^a \end{aligned}$$

  is a one-dimensional representation, which we will denote Det^a. It is a rational representation for every $a \in \mathbb{Z}$, and a polynomial representation if $a \geq 0$.

[1] Algebraic geometers say $GL(n, \mathbb{C})$ is a *linear algebraic group*, differential geometers say it is a *complex Lie group*.

Remark 9.4. Direct sums, tensors products, subrepresentations, and duals of rational representations are again rational. Direct sums, tensors products, and subrepresentations of polynomial representations are again polynomial. But the dual of polynomial representation need not be polynomial: for instance the dual of the representation $A \mapsto \det A$ is the representation $A \mapsto (\det A)^{-1}$.

Remark 9.5. For a rational representation, we require the rational functions $P_{k\ell}(a_{ij})$ to be defined at all points of $GL(n, \mathbb{C})$. One can verify that this implies that $P_{k\ell}(a_{ij}) = \frac{p_{k\ell}(a_{ij})}{\det(a_{ij})^{b_{k\ell}}}$ for some polynomial $p_{k\ell}$ and some $b_{k\ell} \in \mathbb{N}$.

Remark 9.6. * A representation ρ being rational means that ρ is not just a morphism of groups, but also a morphism of algebraic varieties. I.e. ρ is a morphism of linear algebraic groups.

Example 9.7. Choose a basis $\{e_1, \ldots, e_n\}$ of V, so we can write $V = \mathbb{C}^n$ and $GL(V) = GL(n, \mathbb{C})$. We take exterior powers of the standard representation $V = \mathbb{C}^n$. Explicitly: $\bigwedge^d V$ has a basis given by $\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < \ldots < i_d \leq n\}$. If we use this basis to identify $GL(\bigwedge^d V)$ with $GL\left(\binom{n}{d}, \mathbb{C}\right)$, the representation $\bigwedge^d V$ is given by mapping a matrix $A \in GL(n, \mathbb{C})$ to it *d-th compound matrix*, whose entries are the $d \times d$ minors of A.

If take $n = 3$ and $d = 2$, and order the basis of $\bigwedge^2 \mathbb{C}^3$ as $(e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$, we get

$$GL(3, \mathbb{C}) \cong GL(V) \to GL(\bigwedge^2 V) \cong GL(3, \mathbb{C})$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} & a_{12}a_{23} - a_{13}a_{22} \\ a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{33} - a_{13}a_{32} \\ a_{21}a_{32} - a_{22}a_{31} & a_{21}a_{33} - a_{23}a_{31} & a_{22}a_{33} - a_{23}a_{32} \end{pmatrix}$$

Note that taking $d = n$, we have that $\bigwedge^n V$ is one-dimensional, and we recover the determinant representation from the previous example.

Remark 9.8. The first important statements we proved for representations about finite groups were Schur's Lemma (Theorem 3.24) and complete reducibility (Theorem 3.26). We in fact stated and proved Schur's Lemma for an arbitrary group G, so in particular also for $GL(V)$. In contrast, our proof for complete reducibility relied on the fact that G was finite. It turns out that complete reducibility still holds for rational/polynomial representations of $GL(V)$.

THEOREM 9.9. *Let V be a rational representation of $GL(V)$, and let $W \subseteq V$ be a subrepresentation. Then there exists another subrepresentation $W' \subseteq V$ such that $V = W \oplus W'$.*

PROOF. Omitted. □

10. Characters for $GL(n, \mathbb{C})$

In this section, we will once and for all fix a basis of V, i.e. we identify $GL(V)$ with $GL(n, \mathbb{C})$. Analogously to finite groups, we would like to study the representations of $GL(n, \mathbb{C})$ via their characters. Recall that the conjugacy classes in $GL(n, \mathbb{C})$ correspond to Jordan normal forms. In particular there are infinitely many conjugacy classes, so we cannot represent the character of a representation with a finite character table. However, observe that at least for a rational representation W, if we know the value of the character χ_W on the dense subset of *diagonalizable* matrices in $GL(n, \mathbb{C})$, we know it everywhere by a continuity argument. But since the character is constant on conjugacy classes, this means that the character of a $GL(n, \mathbb{C})$-representation is uniquely determined by its value on *diagonal* matrices. This motivates the definition below.

The invertible diagonal matrices form a subgroup of $GL(n, \mathbb{C})$, which we call a *torus* and will denote by T. We have an isomorphism $T \cong (\mathbb{C}^*)^n$.

Definition 10.1. For $\rho_W : GL(n, \mathbb{C}) \to GL(W)$ a representation, the *restricted character* $\chi_W\big|_T$ is the restriction of the character to T, i.e. the composition

$$(\mathbb{C}^*)^n \cong T \hookrightarrow GL(n, \mathbb{C}) \xrightarrow{\mathrm{tr}} \mathbb{C}$$
$$(t_1, \dots, t_n) \mapsto \mathrm{tr}\left(\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} \right).$$

Note that $\chi_W\big|_T$ is a symmetric function in the variables $(t_1, \dots, t_n)$: for every permutation $\sigma \in \mathfrak{S}_n$, we have that $\chi_W\big|_T(t_{\sigma(1)}, \dots, t_{\sigma(n)}) = \chi_W\big|_T(t_1, \dots, t_n)$. If W is a polynomial representation, then $\chi_W\big|_T$ is a symmetric polynomial, and if W is a rational representation, then $\chi_W\big|_T$ is a symmetric Laurent polynomial.

Remark 10.2. For representations of $GL(n, \mathbb{C})$, we will abuse notation and simply write χ_W instead of $\chi_W\big|_T$; we will also just say "character" when we mean "restricted character".

The formulas (4.1), (4.2), (4.4), (4.5) from Proposition 4.4 are still valid, and allow us to compute the character of direct sums, tensor products, exterior powers, and symmetric powers.

Exercise 10.3. If W is a representation of $GL(V)$ and W^* is the dual representation, we have

$$\chi_{W^*}(t_1, \dots, t_n) = \chi_W(t_1^{-1}, \dots, t_n^{-1}).$$

In particular, the dual of a polynomial representation is typically not polynomial.

Example 10.4.

- If $W = V$ is the standard representation, then
$$\chi_V(t_1, \dots, t_n) = t_1 + t_2 + \cdots + t_n.$$
- If W is the determinant representation, then
$$\chi_W(t_1, \dots, t_n) = t_1 \cdot t_2 \cdots t_n.$$

Exercise 10.5. (Exercise 1 on Sheet 5.) Let V be the standard representation of $GL(n, \mathbb{C})$.

- The character of $\bigwedge^d V$ is given by the *elementary symmetric polynomial*
$$E_d(t_1, \dots, t_n) = \sum_{1 \le i_1 < \dots < i_d \le n} t_{i_1} \cdots t_{i_d}.$$
(hint: use Example 9.7).
- The character of $S^d V$ is given by the *complete homogeneous symmetric polynomial*
$$H_d(t_1, \dots, t_n) = \sum_{1 \le i_1 \le \dots \le i_d \le n} t_{i_1} \cdots t_{i_d}.$$

11. Irreducible representations of $GL(V)$

For a finite group G, every irrep is a subrepresentation of the regular representation $\mathbb{C}G$. This is what we used to construct all irreps of the symmetric group. For $G = GL(V)$, the regular representation is not finite-dimensional, so we will use a different strategy. Namely: we will start from the standard representation V, take high tensor powers $V^{\otimes d}$, and show that every irrep of $GL(V)$ occurs as a subrepresentation of some $V^{\otimes d}$.

Consider the $GL(V)$-representation $V^{\otimes d}$. The left action of $GL(V)$ is given by

$$g \cdot (v_1 \otimes \cdots \otimes v_d) = gv_1 \otimes \cdots \otimes gv_d.$$

Note that $V^{\otimes d}$ also comes equipped with a right action of the symmetric group $\mathfrak{S}_d$:

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

Finally, note that these actions commute:

$$\big(g \cdot (v_1 \otimes \cdots \otimes v_d)\big) \cdot \sigma = g \cdot \big((v_1 \otimes \cdots \otimes v_d) \cdot \sigma\big).$$

In the language of Section 5: $V^{\otimes d}$ is an $(\mathbb{C}GL(V), \mathbb{C}\mathfrak{S}_d)$-bimodule.

Definition 11.1. Let $\lambda \vdash d$ be a partition. Let T_λ be a Young tableau with underlying diagram λ and let $c(T_\lambda) \in \mathbb{C}\mathfrak{S}_d$ be the corresponding Young symmetrizer. Then we define the *Schur module*

$$\mathbb{S}_\lambda(V) := V^{\otimes d} \cdot c(T_\lambda) = \{\omega \cdot c(T_\lambda) \mid \omega \in V^{\otimes d}\},$$

viewed as a $GL(V)$-representation.

Remark 11.2. In the definition above we chose a Young tableau $T = T_\lambda$. Similarily to Proposition 8.7, we can see that the representation $\mathbb{S}_\lambda(V)$ we defined does not depend on the chosen tableau. Indeed: if T' were a different tableau on λ, we would have $c(T') = e_\sigma \cdot c(T) \cdot e_{\sigma^{-1}}$ for some permutation $\sigma \in \mathfrak{S}_d$. But then

$$V^{\otimes d} \cdot c(T') = V^{\otimes d} \cdot e_\sigma \cdot c(T) \cdot e_{\sigma^{-1}} = V^{\otimes d} \cdot c(T) \cdot e_{\sigma^{-1}},$$

which is isomorphic to $V^{\otimes d} \cdot c(T)$ via right multiplication with e_σ.

Example 11.3. If $\lambda = (d)$, then $c(T_\lambda) = \sum_{\sigma \in \mathfrak{S}_d} e_\sigma$, independent of the choice of T_λ. We see that $\mathbb{S}_\lambda(V) \subseteq V^{\otimes d}$ is spanned by the vectors of the form

$$\sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}.$$

Comparing with Remark 2.18, we see that $\mathbb{S}_\lambda(V) = S^d V \subseteq V^{\otimes d}$ is precisely the d'th symmetric power of V.

Example 11.4. If $\lambda = (1, \ldots, 1)$ (where there are d ones), then $c(T_\lambda) = \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) e_\sigma$, independent of the choice of T_λ. Now $\mathbb{S}_\lambda(V) \subseteq V^{\otimes d}$ is spanned by the vectors of the form

$$\sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}.$$

Again, comparing with Remark 2.18, we see that $\mathbb{S}_\lambda(V) = \bigwedge^d V \subseteq V^{\otimes d}$ is precisely the d'th alternating power of V.

Example 11.5. Let $\lambda = (2, 1)$, and let T_λ be the Young tableau

$$T=\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array},$$

for Example 8.6. The $c(T) = e_{id} + e_{(12)} - e_{(13)} - e_{(132)}$, so $\mathbb{S}_\lambda(V) \subset V^{\otimes 3}$ is spanned by all vectors of the form

$$v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2.$$

It turns out that, if we restrict ourselves to polynomial representations, the Schur modules form a complete set of nonisomorphic irreps. More precisely, we have the following:

Theorem 11.6. *Write $n = \dim V$.*

1. *The Schur module $\mathbb{S}_\lambda(V)$ is zero if and only if the Young diagram λ has more than n rows.*
2. *$\mathbb{S}_\lambda(V)$ is an irreducible $GL(V)$-representation.*
3. *If λ and μ are distinct Young diagrams with at most n rows, the representations $\mathbb{S}_\lambda(V)$ and $\mathbb{S}_\mu(V)$ are not isomorphic.*
4. *Every polynomial irreducible $GL(V)$-representation is isomorphic to some $\mathbb{S}_\lambda(V)$, where λ is a Young diagram with at most n rows. In other words, we have a one-to-one correspondence*

$$\left\{ \begin{array}{l} \textit{Polynomial irreps of } GL(V) \\ \textit{up to isomorphism.} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}.$$

In the final section of this course, we will prove part (2). We unfortunately won't have time for the other parts. We close this section with a formula for the character of a Schur module (again without proof):

Theorem 11.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition with $k \leq n$ parts. If $k < n$, we define $\lambda_{k+1} = \ldots = \lambda_n = 0$. The character of $\mathbb{S}_\lambda(V)$ is the symmetric polynomial given by*

$$\frac{\det\begin{pmatrix} t_1^{\lambda_1+n-1} & t_1^{\lambda_2+n-2} & \cdots & t_1^{\lambda_{n-1}+1} & t_1^{\lambda_n} \\ t_2^{\lambda_1+n-1} & t_2^{\lambda_2+n-2} & \cdots & t_2^{\lambda_{n-1}+1} & t_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_n^{\lambda_1+n-1} & t_n^{\lambda_2+n-2} & \cdots & t_n^{\lambda_{n-1}+1} & t_n^{\lambda_n} \end{pmatrix}}{\det\begin{pmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{pmatrix}}.$$

This polynomial is known as the Schur polynomial $S_\lambda(t_1, \ldots, t_n)$.

One can show that the Schur polynomials S_λ, where λ runs over all Young diagrams with at most n rows, form a basis for the space of symmetric polynomials in $t_1, \ldots, t_n$. Together with the above theorem, this implies that any polynomial GL_n-representation is uniquely determined by its character. Moreover, decomposing a given representation into irreducibles corresponds to writing a given symmetric polynomial as a linear combination of Schur polynomials.

11.1. Proof that Schur modules are irreducible. The goal of this section is to prove the following:

THEOREM 11.8. *The $GL(V)$-representation $\mathbb{S}_\lambda(V)$ is irreducible.*

In order to prove this theorem, we define $B \subset \operatorname{End}(V^{\otimes d})$ to be the algebra of endomorphisms of U that commute with the $\mathfrak{S}_d$-action:

$$B = \operatorname{End}_{\mathfrak{S}_d}(V^{\otimes d}) = \{\varphi : V^{\otimes d} \to V^{\otimes d} \text{ linear} \mid \varphi(u \cdot g) = \varphi(u) \cdot g \quad \forall u \in V^{\otimes d}, g \in \mathfrak{S}_d\}.$$

The next lemma shows that, for the purposes of our proof, we can think about B-modules instead of $GL(V)$-representations.

Lemma 11.9. *A subspace of $V^{\otimes d}$ is a sub-B-module if and only if it is invariant under $GL(V)$.*

PROOF. We have a natural map $\rho : GL(V) \to GL(V^{\otimes d}) \subset \operatorname{End}(V^{\otimes d})$ encoding the $GL(V)$-action. Since the actions of $\mathfrak{S}_d$ and $GL(V)$ commute, the image of ρ is contained in $\operatorname{End}_{\mathfrak{S}_d}(V^{\otimes d}) = B$. This implies that every subspace of $V^{\otimes d}$ invariant under B is in particular invariant under $GL(V)$. In order show the converse, it suffices to show that $\operatorname{im} \rho$ linearly spans B.

For this, note that there is a natural isomorphism $\operatorname{End}(V^{\otimes d}) \cong \operatorname{End}(V)^{\otimes d}$, and that B consists of all elements that are invariant under the natural right $\mathfrak{S}_d$-action on $\operatorname{End}(V)^{\otimes d}$. Using the notation from Section 2, this means that $B = S^d(\operatorname{End} V)$ is the d'th symmetric power of the space $\operatorname{End} V$. But by Proposition 2.23 we know that this space is linearly spanned by all vectors of the form $w \otimes \ldots \otimes w$, where $w \in \operatorname{End} V$. So we get the following chain of equalities of subspaces of $\operatorname{End}(V)^{\otimes d}$:

$$\begin{aligned} B &= S^d(\operatorname{End} V) \\ &= \operatorname{Span}\{w \otimes \ldots \otimes w \mid w \in \operatorname{End}(V)\} \\ &= \operatorname{Span}\{w \otimes \ldots \otimes w \mid w \in GL(V)\} \\ &= \operatorname{Span} \operatorname{im} \rho, \end{aligned}$$

where the third equality follows since $GL(V)$ is dense in $\operatorname{End}(V)$. □

In particular, the above lemma implies that to show Theorem 11.8, it is equivalent to show that $\mathbb{S}_\lambda(V)$ is an irreducible B-module. We will show this in two steps:

(1) There is an isomorphism

$$\mathbb{S}_\lambda(V) \cong V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda \tag{11.1}$$

of left B-modules, where V_λ is the irrep of $\mathfrak{S}_d$ corresponding to λ.

(2) From the irreducibility of V_λ we can conclude irreducibility of $\mathbb{S}_\lambda(V)$.

Both steps follow from more general statements about group algebras of finite groups:

Lemma 11.10. *Let A be a $\mathbb{C}$-algebra, and $c \in A$ an idempotent. Let U be a finite-dimensional right A-module, and $B = \operatorname{End}_A(U)$. Then the canonical map*

$$\begin{aligned} \varphi : U \otimes_A Ac &\to U \cdot c \\ u \otimes ac &\mapsto u \cdot (ac) \end{aligned}$$

is an isomorphism of left B-modules.

Proof. The map is a morphism of left B-modules by construction. Surjectivity is easy: a preimage of $u \cdot c$ is given by $u \otimes c$. For injectivity, note that every element of $U \otimes_A Ac$ is of the form $u \otimes c$. If $u \otimes c \in \ker(\phi)$ this means that $u \cdot c = 0$. But then in $U \otimes_A Ac$ we have the equality $u \otimes c = u \otimes c^2 = u \cdot c \otimes c = 0$, where we used that c is an idempotent. □

Applying the above lemma to $A = \mathbb{C}\mathfrak{S}_d$, $U = V^{\otimes d}$, and $c = c(T_\lambda)$ yields the desired isomorphism (11.1).

Lemma 11.11. *Let $A = \mathbb{C}G$, where G is any finite group. Let W be an irreducible left G-module, and U be any finite-dimensional right A-module. Write $B = \mathrm{End}_G(U)$. Then $U \otimes_A W$ is an irreducible left B-module.*

Before we go into the proof of this, we need to say a couple of things about *right* $\mathbb{C}G$-modules. Through the entirety of sections Section 3 and Section 4, we could have worked with right group actions instead of left group actions: this leads to the notion of a *right* G-representation. Just as usual (i.e. left) G representations correspond to left $\mathbb{C}G$-modules, right G-representations correspond to right $\mathbb{C}G$-modules. What is more, if V is a left G-representation, we can give V^* the structure of a right G-representation[2] by $\langle \beta \cdot g, v\rangle = \langle \beta, g \cdot v\rangle$. One verifies that V is an irrep if and only V^* is an irrep. So if $\{V_i\}_{i\in I}$ is a complete set of nonisomorphic left irreps of G, then $\{V_i^*\}_{i\in I}$ is a complete set of nonisomorphic right irreps of G.

Proposition 11.12. *[3] *Let V and W be representations of a finite group G. Then the natural linear map*

$$\mathrm{Hom}_G(V,W) \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(V,W) \xrightarrow{\cong} V^* \otimes_{\mathbb{C}} W \twoheadrightarrow V^* \otimes_{\mathbb{C}G} W \tag{11.2}$$

is an isomorphism.

Proof. Consider the linear map

$$V^* \otimes_{\mathbb{C}G} W \xrightarrow{\Phi} \mathrm{Hom}_G(V,W) \tag{11.3}$$

$$\beta \otimes w \mapsto \left(v \mapsto \frac{1}{|G|} \sum_{g\in G} \langle \beta, g^{-1} \cdot v\rangle g \cdot w \right). \tag{11.4}$$

This is well-defined, since $\Phi(\beta \cdot h \otimes w) = \Phi(\beta \otimes h \cdot w)$ and $\Phi(\beta \otimes w)(h \cdot v) = h \cdot \Phi(\beta \otimes w)(v)$ for all $h \in G$ (exercise). We now claim that Φ is the inverse to our morphism (11.2). To see this, we can choose a basis $\{e_1, \ldots, e_n\}$ of V and write (11.2) as

$$\begin{aligned} \mathrm{Hom}_G(V,W) &\xrightarrow{\Psi} V^* \otimes_{\mathbb{C}G} W \\ f &\mapsto \sum_i e_i^* \otimes f(e_i). \end{aligned}$$

[2] This is different from Definition 3.16, where we defined the action slightly differently so it would be a left action. For the rest of this section, V^* will always denote this right representation.

[3] We need this proposition in the proof of Lemma 11.11; the proof was skipped during the lecture and only added later to the notes.

We check that $\Phi \circ \Psi = id$:

$$\begin{aligned}
\Phi(\Psi(f))(v) &= \Phi(\sum_i e_i^* \otimes f(e_i))(v) \\
&= \frac{1}{|G|} \sum_i \sum_{g \in G} \langle e_i^*, g^{-1} \cdot v \rangle g \cdot f(e_i) \\
&= \frac{1}{|G|} \sum_{g \in G} g \cdot f(\sum_i \langle e_i^*, g^{-1} \cdot v \rangle e_i) \\
&= \frac{1}{|G|} \sum_{g \in G} g \cdot f(g^{-1} \cdot v) \\
&= \frac{1}{|G|} \sum_{g \in G} f(g \cdot g^{-1} \cdot v) \\
&= f(v).
\end{aligned}$$

We check that $\Psi \circ \Phi = id$:

$$\begin{aligned}
\Psi(\Phi(\beta \otimes w)) &= \sum_i e_i^* \otimes \left(\frac{1}{|G|} \sum_{g \in G} \langle \beta, g^{-1} \cdot e_i \rangle g \cdot w \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\left(\sum_i \langle \beta \cdot g^{-1}, e_i \rangle e_i^* \right) \cdot g \right) \otimes w \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\beta \cdot g^{-1} \cdot g \right) \otimes w \\
&= \beta \otimes w.
\end{aligned}$$

□

Proof of Lemma 11.11. We can write $W = V_j$. Let us first consider the case where U is irreducible as well, so we can write $U = V_i^*$. By Proposition 11.12, we have an isomorphism

$$V_i^* \otimes_{\mathbb{C}G} V_j \cong \mathrm{Hom}_G(V_i, V_j).$$

But by Schur's lemma, $\mathrm{Hom}_G(V_i, V_j)$ has dimension at most one, in particular it is irreducible.

For the general case, we decompose $U = \bigoplus_{i \in I} (V_i^*)^{\oplus n_i}$. Analogously to the above we have

$$U \otimes_{\mathbb{C}G} V_j \cong \bigoplus_{i \in I} \mathrm{Hom}_G(V_i, V_j)^{\oplus n_i} \cong \mathrm{Hom}_G(V_j, V_j)^{\oplus n_j} \cong \mathbb{C}^{n_j}.$$

Moreover, using Corollary 3.34, we find that

$$B = \mathrm{End}_G(U) = \mathrm{End}_G(\bigoplus_{i \in I} (V_i^*)^{\oplus n_i}) \cong \bigoplus_{i \in I} \mathrm{Mat}(n_i \times n_i, \mathbb{C}).$$

By unwinding the definitions a bit more carefully, we see that the action of B on $U \otimes_{\mathbb{C}G} V_j$ corresponds to the action of $\mathrm{Mat}(n_j \times n_j, \mathbb{C})$ on $\mathbb{C}^{n_j}$ by left multiplication. This clearly has no nontrivial submodules. □

PROOF OF THEOREM 11.8. Applying Lemma 11.11 to $G = \mathfrak{S}_d$, $W = V_\lambda$ and $U = V^{\otimes d}$ and using (11.1) we find that $\mathbb{S}_\lambda(V) \cong V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda$ is an irreducible B-module. By Lemma 11.9 it is also an irreducible $GL(V)$-representation. □

We can go one step further: recall from Proposition 6.6 that we have an isomorphism of algebras

$$\mathbb{C}\mathfrak{S}_d \cong \bigoplus_{\lambda \vdash d} \operatorname{End}(V_\lambda).$$

In particular this is an isomorphism of $(\mathbb{C}\mathfrak{S}_d, \mathbb{C}\mathfrak{S}_d)$-bimodules. But as $(\mathbb{C}\mathfrak{S}_d, \mathbb{C}\mathfrak{S}_d)$-bimodules, we have that $\operatorname{End}(V_\lambda) \cong V_\lambda \otimes_{\mathbb{C}} V_\lambda^*$, where we view V_λ^* as a right $\mathfrak{S}_d$-representation.[4] Now we can write the following chain of isomorphisms of $(\mathbb{C}GL(V), \mathbb{C}\mathfrak{S}_d)$-bimodules:

$$\begin{aligned} V^{\otimes d} &\cong V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} \mathbb{C}\mathfrak{S}_d \\ &\cong V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} \left(\bigoplus_{\lambda \vdash d} \operatorname{End}(V_\lambda) \right) \\ &\cong \bigoplus_{\lambda \vdash d} V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} (V_\lambda \otimes_{\mathbb{C}} V_\lambda^*) \\ &\cong \bigoplus_{\lambda \vdash d} \left(V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda \right) \otimes_{\mathbb{C}} V_\lambda^* \\ &\cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} V_\lambda^*. \end{aligned}$$

This isomorphism

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} \mathbb{S}_\lambda(V) \otimes_{\mathbb{C}} V_\lambda^* \tag{11.5}$$

is known as *Schur-Weyl duality*. It simultaneously gives the decomposition of $V^{\otimes d}$ as a (left) $GL(V)$-representation, and as a right $\mathfrak{S}_d$-representation. In particular, if in (11.5) we forget about the $\mathfrak{S}_d$-action, we get an isomorphism of $GL(V)$-representations

$$V^{\otimes d} \cong \bigoplus_{\lambda \vdash d} (\mathbb{S}_\lambda(V))^{\oplus \dim V_\lambda}.$$

This is the natural generalization of the isomorphism $V^{\otimes 2} \cong S^2 V \oplus \bigwedge^2 V$ from (2.16).

12. * Closing remarks and further directions

12.1. Rational representations and representations of $SL(V)$. Once we understand the polynomial representations of $GL(V)$, it is only a small step to describe the rational representations as well. Recall that polynomial irreps are in bijection with partitions, i.e. tuples $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. To label the *rational* representations, we just need to drop this "≥ 0".

Proposition 12.1. *Write $n = \dim V$. For $\lambda = (\lambda_1, \dots, \lambda_k)$ a partition with at most n rows, we write $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_i = 0$ for $k < i \leq n$.*

[4] This is a special case of the following: if A, B are $\mathbb{C}$-algebras, V is a left A-module and W is a left B-module, then V^* is naturally a right A-module, $\operatorname{Hom}_{\mathbb{C}}(V, W)$ is naturally a (B, A)-bimodule, and the isomorphism $\operatorname{Hom}_{\mathbb{C}}(V, W) \cong W \otimes_{\mathbb{C}} V^*$ from Example 2.13 is an isomorphism of (B, A)-bimodules.

(1) *Every irreducible rational representation of $GL(V)$ is isomorphic to $\mathbb{S}_\lambda(V) \otimes \mathrm{Det}^a$ for some partition λ with at most n rows and some $a \in \mathbb{Z}$.*
(2) *For $\lambda = (\lambda_1, \dots, \lambda_n)$, the representation $\mathbb{S}_\lambda(V) \otimes \mathrm{Det}$ is isomorphic to $\mathbb{S}_{\lambda'}(V)$, where $\lambda' = (\lambda_1 + 1, \dots, \lambda_n + 1)$.*
(3) *There is a one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{Rational irreps of } GL(V) \\ \text{up to isomorphism.} \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \{(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n \mid \mu_1 \geq \dots \geq \mu_n\}$$

given by mapping $\mathbb{S}_\lambda(V) \otimes \mathrm{Det}^a$ to the tuple $(\lambda_1 + a, \dots, \lambda_n + a)$.

Proof sketch. Part (1) follows from Remark 9.5: write the representation W as a map $GL(n, \mathbb{C}) \to GL(m, \mathbb{C})$ given by rational functions $P_{k\ell}(a_{ij}) = \frac{p_{k\ell}(a_{ij})}{\det(a_{ij})^{b_{k\ell}}}$ where $p_{k\ell}$ is a polynomial and $b_{k\ell} \in \mathbb{N}$. Write $b = \max\{b_{k\ell}\}$, then $W \otimes \mathrm{Det}^b$ is a polynomial irrep, hence $W \otimes \mathrm{Det}^b \cong \mathbb{S}_\lambda(V)$ for some λ, hence $W \cong \mathbb{S}_\lambda(V) \otimes \mathrm{Det}^{-b}$.

Part (2) can be seen by computing the characters of both sides using Theorem 11.7. Part (3) then follows from the first two parts. □

Remark 12.2. Closely related to $GL(V)$ is the *special linear group*

$$SL(V) = \{g \in GL(V) \mid \det(g) = 1\}.$$

For $SL(V)$, there is no distinction between rational and polynomial representations. Every rational irrep of $GL(V)$ restricts to a rational irrep of $SL(V)$. Moreover, every rational irrep of $SL(V)$ arises in this way, and two irreps of $GL(V)$ restrict to the same irrep of $SL(V)$ if and only if they agree up to tensoring with a power of Det. From this we can get a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{Rational irreps of } SL(V) \\ \text{up to isomorphism.} \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{(\mu_1, \dots, \mu_{n-1}) \in \mathbb{Z}^{n-1} \mid \mu_1 \geq \dots \geq \mu_{n-1} \geq 0\right\}.$$

12.2. Decomposing tensor products. One question that often arises in practice is the following:

> Given two $GL(V)$-irreps $\mathbb{S}_\lambda(V)$ and $\mathbb{S}_\mu(V)$, what is the decomposition of the tensor product $\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V)$?

In other words, we want to find the coefficients $N_{\lambda\mu}^\nu$ in the decomposition

$$\mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V) \cong \bigoplus_\nu \mathbb{S}_\nu(V)^{\oplus N_{\lambda\mu}^\nu}.$$

By the results from the previous section, this amounts to taking a product of Schur polynomials and writing it in the Schur basis:

$$S_\lambda(t_1, \dots, t_n) \cdot S_\mu(t_1, \dots, t_n) \cong \sum_\nu N_{\lambda\mu}^\nu S_\nu(t_1, \dots, t_n).$$

The coefficients $N_{\lambda\mu}^\nu$ are known as the *Littlewood-Richardson coefficients*; there is a combinatorial formula for them known as the *Littlewood-Richardson rule*, see for instance [**FH91**, Appendix A].

12.3. Omitted proofs. In Chapters 3 and 4, several proofs have been omitted. Most of these can be found in Lectures 4, 6 and Appendix A of [**FH91**]. One important exception is part 4 of Theorem 11.6, which states that the Schur modules are in fact all polynomial irreps of $GL(V)$. The approach taken in [**FH91**] to show this goes via Lie algebras (see below), and the proof is only completed in Lecture 15. For a proof that is closer to the approach taken in this lecture, see [**CB90**].

12.4. Linear algebraic groups. There are two ways to place our study of $GL(n,\mathbb{C})$ and its representations in a more general context. If you like algebraic geometry, the natural definition to make is the following.

Definition 12.3.
- An *affine algebraic variety* is a subset of $X \subseteq \mathbb{C}^n$ of the form $X = \{x \in \mathbb{C}^n \mid f_1(x) = \ldots = f_k(x) = 0\}$, where $f_1, \ldots, f_k$ are polynomials in n variables.
- A *morphism between affine algebraic varieties* $X \subseteq \mathbb{C}^n$ and $Y \subseteq \mathbb{C}^m$ is a map $\varphi : X \to Y$ that can be described by polynomials: i.e. there exist $p_1, \ldots p_m$ in n variables such that $\varphi(x) = (p_1(x), \ldots, p_m(x)) \in Y \subseteq \mathbb{C}^m$ for each $x \in X$.
- An *affine algebraic group* or *linear algebraic group* (over $\mathbb{C}$) is an affine algebraic variety G equipped with the structure of a group, such that the multiplication map $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$ are morphisms of algebraic varieties.
- A *morphism of affine algebraic groups* is a map $G \to H$ that is both a morphism of affine varieties and a morphism of groups.

From the above definition, it is clear that $SL(n,\mathbb{C})$ is an affine algebraic variety, and hence an affine algebraic group: we have

$$SL(n,\mathbb{C}) = \{M \in \mathrm{Mat}(n \times n, \mathbb{C}) \mid \det(M) - 1 = 0\} \subseteq \mathrm{Mat}(n \times n, \mathbb{C}) \cong \mathbb{C}^{n^2}$$

To view $GL(n,\mathbb{C})$ as an affine algebraic variety, we need a little trick:

$$GL(n,\mathbb{C}) = \{(M, D) \mid \det(M) \cdot D - 1 = 0\} \subseteq \mathrm{Mat}(n \times n, \mathbb{C}) \times \mathbb{C} \cong \mathbb{C}^{n^2+1}.$$

This agrees with the usual definition of $GL(n,\mathbb{C})$: identify an invertible matrix M with the pair $(M, (\det M)^{-1})$.

Remark 12.4. In fact, it turns out that every affine algebraic group is a closed subgroup of some $GL(n,\mathbb{C})$. Some other examples of affine algebraic groups include the additive group $(\mathbb{C}, +)$, the orthogonal and special orthogonal groups $O(n,\mathbb{C})$ and $SO(n,\mathbb{C})$, and the symplectic group $Sp(n,\mathbb{C})$.

Definition 12.5. An *algebraic representation* of a linear algebraic group G is a morphism $G \to GL(m,\mathbb{C})$ of linear algebraic groups.

In the case $G = GL(n,\mathbb{C})$ or $G = SL(n,\mathbb{C})$, this agrees with our definition of rational representation.

12.5. Lie groups and lie algebras. The differential-geometric viewpoint is to see $GL(n,\mathbb{C})$ not as a linear algebraic group, but as a complex manifold. This leads to the notion of a *Lie group.* The remainder of our main reference [**FH91**] is devoted to the representation theory of Lie groups and Lie algebras. I will here attempt to give a very brief idea of what this is about and how it relates to what we have been doing.

Definition 12.6. A *complex Lie group* is a complex manifold G equipped with the structure of a group, such that the multiplication map $\mu : G \times G \to G$ and the inversion map $\iota : G \to G$ are morphisms of complex manifolds. A *morphism of complex Lie groups* is a map $G \to H$ that is both a morphism of complex manifolds and a morphism of groups.

Remark 12.7. If in the above we replace "complex manifold" with "smooth manifold", we obtain the definition of a *real Lie group*.

It turns out that an affine algebraic variety is always smooth[5], and therefore a complex Lie group. In particular, $SL(n, \mathbb{C})$, $GL(n, \mathbb{C})$, and the examples from Remark 12.4 are complex Lie groups.

Definition 12.8. A *Lie group representation* of a complex Lie group G is a morphism $G \to GL(n, \mathbb{C})$ of complex Lie groups.

Remark 12.9. Every morphism of linear algebraic groups is a morphism of Lie groups, but not the other way around. In particular, for G a linear algebraic group, there can be Lie group representations of G that are not algebraic. For instance, take $G = (\mathbb{C}, +)$ and the map $G \to GL(1, \mathbb{C}) : z \to e^z$. However, as we will see below, for $G = GL(n, \mathbb{C})$ every Lie group representation is in fact algebraic.

Given a Lie group G, one can take the tangent space T_eG at the identity element, which is just a vector space. It turns out that the group operation on G endows T_eG with an operation $[\cdot, \cdot] : T_eG \times T_eG \to T_eG$ called the *Lie bracket*. This gives T_eG the structure of a so-called *Lie algebra*, which is typically denoted $\mathrm{Lie}(G)$ or simply $\mathfrak{g}$.

Definition 12.10. A *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[X, X] = 0$ for all $X \in \mathfrak{g}$,
- $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

It would take me a bit too much time to explain the geometric construction of the Lie bracket on $\mathrm{Lie}(G)$ from the group structure of G. However, I can tell you what the Lie bracket is for every Lie algebra you will ever encounter: For $G = GL(n, \mathbb{C})$, we have $T_eG = \mathrm{Mat}(n \times n, \mathbb{C}) =: \mathfrak{gl}_n$. The Lie bracket in this case is the commutator: $[X, Y] = XY - YX$. More generally, if G is a closed subgroup of $GL(n, \mathbb{C})$ (for instance $SL(n, \mathbb{C})$, $SO(n, \mathbb{C})$, $\ldots$), then there is a natural inclusion $T_eG \subset \mathrm{Mat}(n \times n, \mathbb{C})$, and the Lie bracket is still given by the commutator.

Definition 12.11. A *Lie algebra representation* is a morphism $\mathfrak{g} \to \mathfrak{gl}_n$ of Lie algebras; i.e. a linear map $\mathfrak{g} \to \mathrm{Mat}(n \times n, \mathbb{C})$ such that

$$\varphi([X, Y]) = \varphi(X) \cdot \varphi(Y) - \varphi(Y) \cdot \varphi(X).$$

THEOREM 12.12. *If G is a simply connected Lie group, then there is a one-to-one correspondence between Lie group representations of G and Lie algebra representations of* $\mathrm{Lie}(G)$.

If we want to apply this to our favorite group $G = GL(n, \mathbb{C})$, we run into the problem that $GL(n, \mathbb{C})$ is not simply connected. However, $SL(n, \mathbb{C})$ is simply connected. After spending some time analyzing the Lie algebra $\mathfrak{sl}_n$, one can show

THEOREM 12.13. *Every representation of $\mathfrak{sl}_n$ is isomorphic to $\mathbb{S}_\lambda(\mathbb{C}^n)$, for λ a partition with at most $n - 1$ parts.*

Then Theorem 12.12 implies that every Lie group representation of $SL(n, \mathbb{C})$ is isomorphic to $\mathbb{S}_\lambda(\mathbb{C}^n)$, foe λ a partition with at most $n-1$ parts. This then implies the corresponding result for $GL(n, \mathbb{C})$ with relatively little extra effort.

[5]This follows from the fact that G acts transitively on itself by left multiplication. So if G had a singularity then every point would be singular, which cannot happen.

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