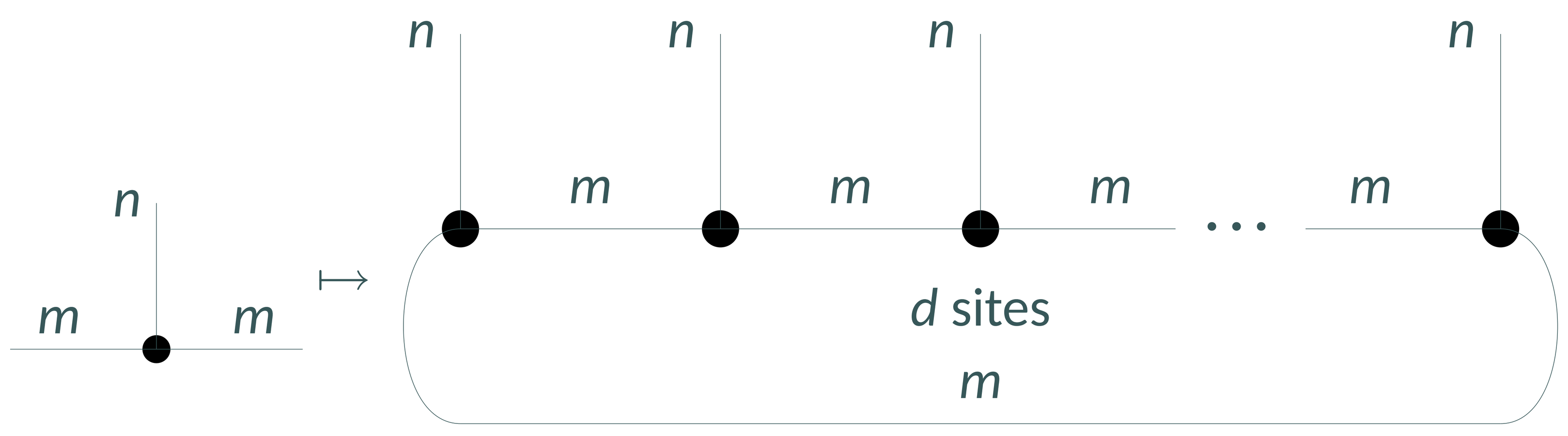
The linear span of uniform matrix product states

Claudia De Lazzari, Harshit J. Motwani, **Tim Seynnaeve**

Uniform matrix product states

The uniform matrix product state parametrization is given by the map

$$\varphi : (\mathbb{C}^{m\times m})^n \to (\mathbb{C}^n)^{\otimes d}$$
$$(A_1,\dots,A_n) \mapsto \sum_{1\le i_1,\dots,i_d\le n} \mathrm{Tr}(A_{i_1}\cdots A_{i_d})\, e_{i_1}\otimes\cdots\otimes e_{i_d}. \tag{1}$$



The variety $\mathrm{uMPS}(m,n,d)$ is the closure of the image of this map.

$$\mathrm{uMPS}(m,n,d) = \overline{\varphi((\mathbb{C}^{m\times m})^n)}$$

Motivation

- They model quantum-mechanical systems of d sites placed on a ring.
- For $m = 1$, we recover the Veronese embedding. Hence $\mathrm{uMPS}(m,n,d)$ is a "noncommutative Veronese variety".

Observations

- The variety $\mathrm{uMPS}(m,n,d)$ is contained in the space $\mathrm{Cyc}^d(\mathbb{C}^n)$ of cyclically symmetric tensors.
- For the case $m = n = 2$, we even have
$$\mathrm{uMPS}(2,2,d) \subseteq \mathrm{Dih}^d(\mathbb{C}^2),$$
where Dih means dihedrally symmetric.
- The variety $\mathrm{uMPS}(m,n,d)$ is invariant under the natural action of GL_n on $(\mathbb{C}^n)^{\otimes d}$.

Invariant theory of matrices

- Let $A_1,\dots,A_n$ be $m\times m$ matrices with generic entries. The ring generated by all polynomials $\mathrm{Tr}(A_{i_1}\cdots A_{i_d})$ is the *trace algebra* $\mathscr{C}_{m,n}$.
- **Fact:** $\mathscr{C}_{m,n}$ consists of all polynomials in the entries of the A_i that are invariant under simultaneous conjugation.
- **Fact:** $\mathscr{C}_{2,2}$ is generated by $\mathrm{Tr}(A_1), \mathrm{Tr}(A_2), \mathrm{Tr}(A_1^2), \mathrm{Tr}(A_1A_2), \mathrm{Tr}(A_2^2)$.
- **Corollary:** get an upper bound
$$\dim\langle \mathrm{uMPS}(2,2,d)\rangle \le \frac{1}{192}d^4 + \text{l.o.t.}$$

Goal

- Determine the linear span of $\mathrm{uMPS}(m,n,d)$; i.e. the smallest vector subspace of $(\mathbb{C}^n)^d$ containing $\mathrm{uMPS}(m,n,d)$.
- Equivalently: given n generic $m\times m$ matrices $A_1,\dots,A_n$, what *linear* relations hold between the d-fold traces $\mathrm{Tr}(A_{i_1}\cdots A_{i_d})$.

Representation theory

- The linear span $\langle \mathrm{uMPS}(m,n,d)\rangle$ is naturally a representation of GL_n.
- To determine its decomposition into irreducibles, it suffices to find the dimension of the *weight spaces*.
- Weight spaces are obtained by fixing $(i_1,\dots,i_d)$ up to permutation in (1).

Finding linear relations

Example: For any 2×2 matrices A_0, A_1, A_2, A_3 and any $k \ge 0$, the following identity holds:

$$\mathrm{Tr}(A_1A_2A_0A_3A_0^k) + \mathrm{Tr}(A_2A_3A_0A_1A_0^k) + \mathrm{Tr}(A_3A_1A_0A_2A_0^k) = \mathrm{Tr}(A_1A_0A_2A_3A_0^k) + \mathrm{Tr}(A_2A_0A_3A_1A_0^k) + \mathrm{Tr}(A_3A_0A_1A_2A_0^k).$$

Proof. We first show the identity for $k = 0, 1$:

$$\mathrm{Tr}(A_1A_2A_0A_3) + \mathrm{Tr}(A_2A_3A_0A_1) + \mathrm{Tr}(A_3A_1A_0A_2) = \mathrm{Tr}(A_1A_0A_2A_3) + \mathrm{Tr}(A_2A_0A_3A_1) + \mathrm{Tr}(A_3A_0A_1A_2)$$
$$\mathrm{Tr}(A_1A_2A_0A_3A_0) + \mathrm{Tr}(A_2A_3A_0A_1A_0) + \mathrm{Tr}(A_3A_1A_0A_2A_0) = \mathrm{Tr}(A_1A_0A_2A_3A_0) + \mathrm{Tr}(A_2A_0A_3A_1A_0) + \mathrm{Tr}(A_3A_0A_1A_2A_0).$$

For $k > 1$, we can write A_0^k as a linear combination of A_0^j for $j < k$; so we can proceed by induction. □

The above linear relation can be generalized to $m\times m$ matrices:

$$\sum_{\sigma\in\mathfrak{S}_{m+1}} \mathrm{sgn}(\sigma)\,\mathrm{Tr}(A_0B^{\sigma(0)}A_1B^{\sigma(1)}\cdots A_{m-1}B^{\sigma(m-1)}A_mB^{\sigma(\ell)}) = 0.$$

By an appropriate substitution, we find:

Theorem

If $n \ge 3$ and $d \ge \frac{(m+1)(m+2)}{2}$, then $\mathrm{uMPS}(m,n,d)$ is contained in a proper linear subspace of the space of cyclically invariant tensors.

Outlook

- Based on computer experiments, we conjecture an exact formula:
$$\dim\langle \mathrm{uMPS}(2,2,d)\rangle = \begin{cases} \frac{1}{192}(d^4 - 4d^2 + 192d + 192) & \text{for } d \text{ even},\\ \frac{1}{192}(d^4 - 10d^2 + 192d + 201) & \text{for } d \text{ odd}.\end{cases}$$
- Can we further exploit invariant theory of matrices?

References

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Contact Information

Harshit J Motwani

- Web: https://sites.google.com/view/harshitjmotwani/home
- Email: HarshitJitendra.Motwani@ugent.be

Tim Seynnaeve

- Web: https://mathsites.unibe.ch/seynnaeve/
- Email: tim.seynnaeve@math.unibe.ch